Agricultural Economics

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Chapter 1. Definitions and Methodology

Chapter 1. Definitions and Methodology :

A. Introduction :

Agricultural is, still, one of the leading economic activities in Egypt. Egyptian agriculture, including production, marketing, and processing, amount for over 40 % of overall Gross Domestic Product (GDP) and over 50 % of total employment. The agricultural exports accounts to 25 % of the total exports and the sector ensure food security a long with political dominance to Egypt. Agricultural, in general, is an open industry which normally affected by all known and unknown conditions. The overall environmental conditions show that agriculture is practiced with intensive irrigation in a hot, arid climate. The significance of agricultural activities to Egypt along with the environmental, social and political conditions justify the importance of studying the economies of agriculture.

This chapter is an introductory chapter which details the main definitions and methodology. The reader my find some sort of redundancy when he reads this chapter. Yet the information as stated hence after is still important to be reemphasized.

A.1. Definitions : Most important in an elementary course is to define the subject. Main definitions are as follows :

1. Agricultural Economics : Theoretically, the agricultural economics is an application of the rules set in economic theory on the agricultural activities. But, in practice, the economics of agriculture is extended to cover all exogenous and endogenous factors which impacts the agriculture. These factors include economic, social, political, and environmental conditions. The comparative advantage of agricultural activities is usually weighted given all of these conditions. The agriculture, as an international industry, is affected by the macroeconomic policies, policies of International Monetary Fund, World Bank, World Trade Organization, and all other economic and

political organization. For instance the agricultural activities is new affected by wars, destruction of old USSR and other comminutes countries. It is affected by policies of developed countries and dominance of USA over the market place. In addition to environmental hazards which also is an important exogenous factor that influence the agricultural production. Earthquakes, winds, difference in temperature between days and nights, seasons, ... etc. The economics of agriculture is also impacted by social variables and quality of social environment. For this reason, we consider the agricultural economics as social science.

2. Farm and Technical Unit : A farm is a production space. The main features of this space is determined by both the fertility and location. David Ricardo had set his theory about the rental-values based on both the fertility and location. The farm may be one piece or more than one piece but under all conditions it should be directed under same management.

In analysis, the farm is a unit of the agricultural industry. The technical unit could be defined as the smallest unit for analysis such as a cow-or feddan of cultivated land. The farm is the unit of production. Data in Tables (1) – (2) show the equivalent of the common measurement units.

Table (1). Land Measurement Units.

Unit	M^2	Unit	M^2
Feddan	4,200	Donam	2,000
Hectar	10,000	Kiratt	183
Acre	4,000		

Table (2). Field Crops Measurement Units

Commodity	Unit	Kilogram	Commodity	Unit	Kilogram
Cotton Unginned	Metric Kantar	157.5	Fenugreek	Ardab	155
Ginned Cotton	Metric Kantar	50	Lupine	Ardab	150
Cotton Seed	Metric Ardab	120	Linseed	Ardab	122
Wheat	Ardab	150	Dry Beans	Ardab	120
Beans	Ardab	155	Dry Peas	Ardab	160
Crushed Beans	Ardab	144	Barley	Ardab	120
Groundnuts	Ardab	75	Maize	Ardab	140
Lentils	Ardab	160	Rice	M. Ton	1,000

Source : M.A.L.R. Economic Affairs Sector, Central Administration of Agriculture Economics. General Dept. of Agric. Statistics, June 1997.

A.2. Structure of Egyptian Agricultural Sector :

Three major activities constitutes the function of the sector. They are :

1. Plant production
2. Animal production
3. Fishing

For each activity, there are sub-sectors such as Horticulture, field crops, poultry, ... etc. Data in tables (3) - (4) clarify the relative importance for each sector and sub-sectors in terms of cultivated area and net income originated in each sub-sector.

Table (3) : Cropped and Cultivated Area in 1997.

Season	Area (Feddan)	% of the total
Total area for winter crops	6205924	44.87
Total area for summer crops	5951588	43.04
Total area for Nili crops	618718	4.47
Total area of Gardens	897821	7.14
Total area of Dates	64979	0.47
Total cropped area	13829030	
Crop Intensification Rate	1.83	100
Total land area	7556847	-

Source : Ministry of Agriculture and Land Reclamation, Agricultural Economic Statistics. 2nd part, issues June 1997 and August 1998.

Table (4). Value of Agricultural Production and Net Income in Current Prices in 1996.

Item	Values 000' L.E.	% of the total
A. Plant production		
A.1. Total value of production	38046079	67.84
A.2. Value of inputs	4974512	8.87
A.3. Net income	33071567	58.97
B. Animal production		
B.1. Total value of production	15470468	27.59
B.2. Value of inputs	8863962	15.81
B.3. Net income	6606506	11.78
C. Fish production		
C.1. Total value of production	2564100	4.57
C.2. Value of inputs	352528	0.63
C.3. Net income	2211572	3.49
Total (A.1. + B.1. + C.1.)	56080647	100
Total (A.2. + B.2. + C.2.)	14191002	25.31
Total (A.3. + B.3. + C.3.)	41889645	74.69

Source : MALR, Agricultural Economic Statistics. Different Issues, 1997.

Accurate look at these statistics implies the dominance of plant production as major land and water user, and as income generator. Differences exist within the plant production between winter and

summer crops, field crops and horticultural, ... etc. Yet the overall view emphasizes the dominance of this sector.

A.3. Current State of Agricultural Sector :

The total land supply in Egypt is almost 7.8 million feddan in 1999. About 5.9 million feddan of these lands are old land. And 1.9 million feddan are the reclaimed lands since 1952. The crop intensification rate is 1.83. Therefore, the total cropped area is about 13.83 million feddan. The land supply is still limited to about 4 % of the total physical land endowment which justifies the importance of maintaining the existing land supply and significance of vertical expansion. GOE plan for year 2017 is to extend the land supply to 25 % of the total area.

A.3.1. Technical Production Coefficients :

For each technical unit, there is limits for better utilizing the resources. For instance, resources, overall, are better utilized at intensification rate about 1.83 which means that the agricultural land is almost used twice a year. Data in table no. (3) support this result. At the small level, the technical production coefficient consider the exact technical level of resources needed to reach maximum output. This definition will help for better understanding of the production function latter on. Economist's views differ from Subject Matter Specialists (SMS's) at this point this because of the fact which states what is technically optimum is not necessarily to be economically optimum. Daniel Bromdey had tried to define a model which utilize the set of

agricultural resources to be technically and economically optimum. Discussion in other parts of this course will clearify this point.

Technical and economic optimality determine the better utilization of the resources. It is vital to the agricultural policy makers to know how to allocate the resources to reach optimality. For instance if we consider one feddan of land which crop would be the best to cultivate to benefit both the farmer and society. Certain answers my exist. But the best answer is to look at the value added and to reallocate the resources according to maximum value added. For instance in Egypt both rice and sugarcane utilize 35 % of the total water supply for agricultural use. At the same time both crops are cultivated in 12 % of cropped area and generate 13 % of the value added from agricultural. This surprising fact set a very big question mark on the agricultural decision. Detailed study to this point by the World Bank experts showed that the total requirement - water / land requirement (m^3/feddan) - for a feddan of sugarcane is 12,000 m^3 of water. The value added per m^3 is L.E. 0.01. The same values for sugar beat is 2700 m^3 of water per feddan and value added is L.E. 0.04. For some vegetable and fruit crop's, the value added increases to limit of L.E. 0.08. But the calculations up to this level is still one sided. In policy evaluation, certain sets of variable should be considered to avoid the side effects. Sometimes the decision is based on cost / benfit and sometimes we consider the cost-effectiveness alternative.

A.3.2. Level of resource - Use :

As stated before, the total land supply is 7.8 million feddan cultivated almost two times a year to reach crop area about 13.83

million feddan. The first degree land supply is limited to 360 thousand feddans which represent about 6.1 % of the total land supply. Second and third land degree are about 84 % of the total land supply. The low quality land, forth degree, is about 10 % of the total land supply. These figures show the importance of improving the quality of land supply through effective maintenance programme.

Total water supply available for agricultural from all sources is 42.70 billion quibic meters. Almost 95 % of the agricultural water requirements is from Nile River - The Gift of God to Egypt - This amounts to 34 billion m^3 or 56 % Nile annual flow. Surface water supply is 1.73 billion m^3 flows from about 9749 wells. This amount represents 3.22 % of the total water supply. The rest of water supply is from treated water and rains. These figures also emphasize the nature of irrigated agricultural. Professor Gamal Hemdan had defined the whole system which require a governorate and a system for resource use. Worthnoting is that 63 % of the total water supply is utilized by summer crops.

Total fixed investment allocated to agriculture is about 7.61 % of the total fixed investment. Total credit supply through principal Bank for Development and Agricultural Credit (BPDAC) is L.E. 2,43 billion in 1989/1990. The total foreign assistant to agriculture since 1981 is about USD 1.0 billion. A long with personal saving, this figures represent the total capital supply to agriculture. The supply is limited as compared to the objective of achieving 3 % - 4 % annual growth in agriculture.

Rural areas is initially labor suppliers this due to the fact that 50 % of the total population is in rural areas. The agricultural workers are the part of labor force which has regular work for about 1800 work hours / year. I do believe that the shortage is seasonal phonemen. But the needs also exist for good quality of training programmes and improvement of the quality of social environment.

Management is practiced in agriculture by farmers with support of cooperatives and extension systems. The rural women is now engaged in management of both the farm and rural house. As sated before for labor, also, the needs for good quality of training programme is vital to Egyptian agriculture and to Egypt. This part of labour force are the means of development. They also are the elements of national dominance since the population is increasing at rate 2.1 % and food supply is limited to almost 2 % rate of growth. Agricultural development should have the first priority in our set of objectives. The situation now in year 2000 imputes certain emergency singes. The agricultural file is the hardest in all foreign trade agreements between Egypt and EC and even between EC and USA and other countries. Now in year 2000 we hear about banana war between European common market and USA, problems considering potato production in Egypt, bans on meat and chicken exports ... etc.

A.3.3. Economic and Technical Efficiency :

Sometimes, one can imagine that mixing the resources in production process is random or at the most purely technical. This wrong views set responsibility on economists. Define, the Production Possibility Fornteer (PPF) to show the maximum outputs (y_i & y_j) produced from a given resources, Figure (1).

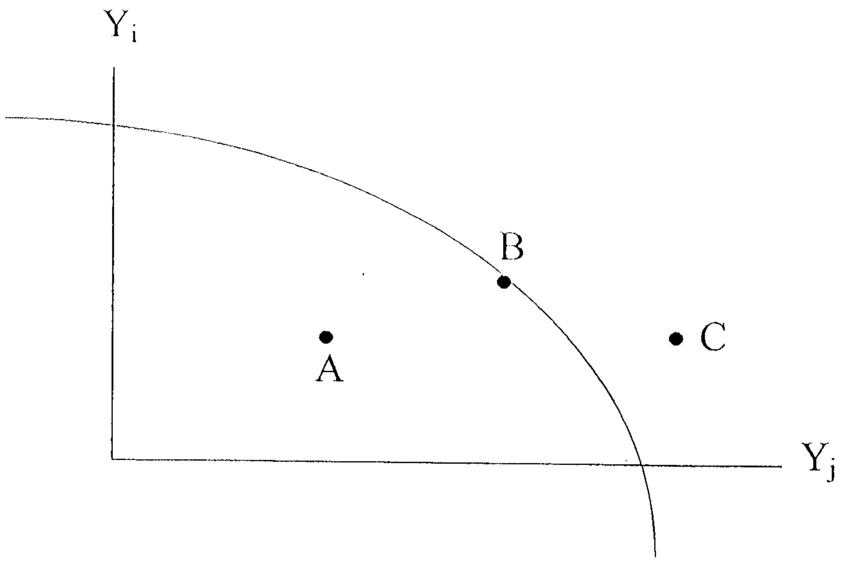


Figure (1) : Production Possibility Fornteer (PPF)

Technicians may recommend producing at (A) or (C) but economist always recommend production at B. This sense implies objection to over (under) - use. The resources, is then, should be allocated in the way that grantee economic efficiency. The government fund and facilitate investment to motivate development through vertical and horizontal agricultural policy programmes. The set of resource supply may grantee technical efficiency. But to increase production from the same level of resource - use this what we mean by economic efficiency. Figure (2). In other words, the resource base produces more output.

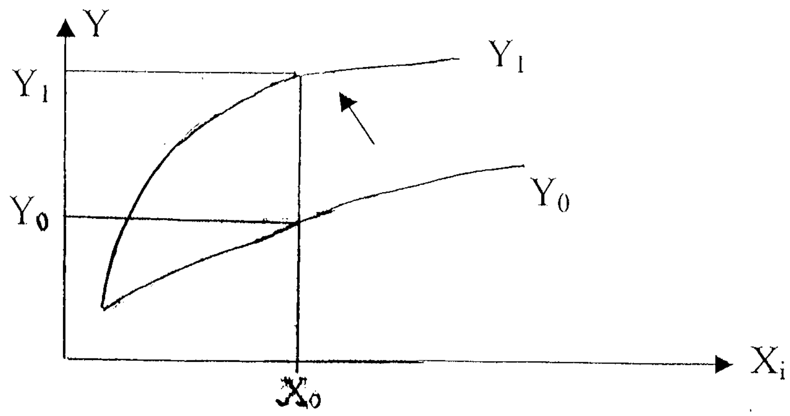


Figure (2). Efficiency from Resource Base.

B. Methodology of Studying Agricultural Economics :

Agricultural Economics is an applied social science. Agricultural Economics is science since its rules are from the economics theory. The agricultural economics is also an applied science since it finds solutions to agricultural problems. The agricultural economics is a social since due to its overall consideration of the farmer's social environment.

B.1. Nature of Agricultural Production :

Agriculture is an "Open" industry. It differs from other industries in nature. The production in Agricultural is normally affected by all exogenous factors starting from soil and whether conditions up to pricing in the international markets. Agricultural problems; however, differs, now, at the beginning of 21 st century between developed and underdeveloped economies. But, in general, the nature of agricultural products, as low price elasticity commodities, has many implications. Set of parameters are estimated in this light. In addition, many hypotheses could also be tested. For instance, net farming income in agriculture fluctuated more in agriculture as compared to industry. The resulted risk affect the production conditions which forced many governments to subsidies agriculture in different forms. Evan in international trade, now, the identified conditions in all files are easy to go; except, those related to agricultural files. Bans, source of embargoes, conditions on production, and other forms of interference are now, in year 2000 practiced in trading the agricultural commodities.

B.1.1. Major Characteristics of Agricultural Production :

a.1. Seasonality : Agricultural activities are seasonal. As stated before in Table (3), the resources are used in three major seasons. This is exactly what we usually mean by word seasonality. The seasonality is due to the open biological nature of agriculture. Seasonality leads to certain results. They are :

a.1.1. Seasonality of Employment and Wages. The labor force is usually faces periods of bottlenecks (Shortages) and surplus. The excess supply of labor (N^s - $N^d \geq 0$) usually exists when nominal wages exceed the equilibrium wage rate. Income increasing and laborers may supply more units until the nominal wages fall to equilibrium point. The opposite exists when the nominal wages are below the equilibrium wage rate. When labor is in excess supply workers are in good standing - better-off - in other cases they are wose-off.

a.1.2. Income seasonality : Income seasonality is a results of seasonality of production. It was known in Egypt before when cotton was the white gold, the farmers - producers - were postponing all their activities, and farm decision until they market the cotton. The implication of income seasonality are seasonlity in consumption and investment. Also high degree of risk are associated with seasonality of production and hence income. In next parts, this point will be clarified.

a.1.3. Seasonality of Related Activities : All other activities in industry, trade, and service sectors related to (or) interacted with agriculture are characterized by seasonality.

Storage and processing are two marketing functions that can help to limit the impacts of seasonality. These functions can help in smoothing the supply over the year to meat the demand at different time periods. Improving such services leads to staple supply on one hand, and keep the farming income at reasonable level on the other hand.

a.2. Competitive Structure of Agriculture : Agriculture in Egypt always practiced in small units. The large number of farms with small homogenous products, a long with free entry and exist, grantee the existence of competition. Even when agricultural sector was under direct control of the government, agricultural had some degree of competition since a single farmer can not affect the production decision or market price. Therefore, he is a price-taker. Three important results of dominance of competitive structure exist. They are :

a.2.1. Since the agricultural industry is competitive, the full cost price is at the point (A) in the short-run. This break-even point is an equilibrium point for profit maximizing firms in a competitive structure. Below this point losses are greater than the fixed costs and firms shutdown at, point (B).

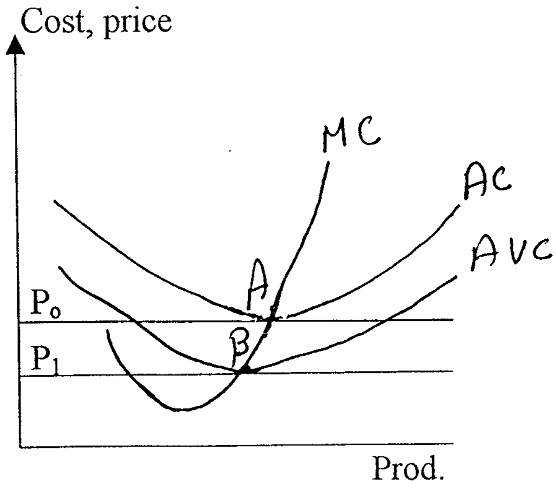


Figure (3). Equilibrium of profit maximizing firms in competitive structure

These result implies that price for a profit maximizing firm in a competitive structure is at $P \geq$ Min. AC = MC. The same results also exist in the long-run as long as the firms are within a competitive industry.

a.2.2. Competitive structure will lead to maximum efficiency, point (B) in Figure (1). Free entry and exist will leave the industry with firms which realize normal profits. Others are shutdown.

a.2.3. Also competitive structure will lead to optimum resources use and allocation.

As in Figure No. (4). The firms in a competitive structure are able the equate :

$$MRT_r = MRS_{Yi,Yj} = -\frac{P_i}{P_j}$$

At this pint we reach optimality and both the producer's and consumer's desires are satisfied. The whole society will be better-off.

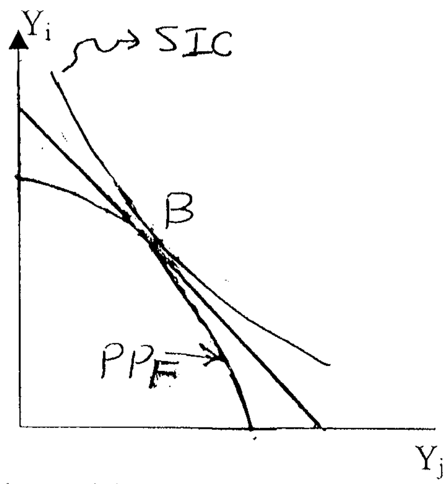


Figure (4). Production Efficiency

a.3. Importance of Land in Production and High Fixed Capital :

Land, structure, equipment represent more than 75 % of the total fixed capital. Fixed costs associated with fixed capital are also high in agriculture as compared to industry. Higher fixed cost leads to higher costing industry and low returns. This fact also implies high degree of risk; especially, in crisis period. In addition, high costing has left the

agriculture with low returns on fixed costs which usually at 2 % - 3 % in Egypt. This low returns justify the needs for subsidizing this sector if returns fall below this moderate level.

a.4. Low Price Elasticity : As you know, the price elasticity of demand (supply) is

$$\frac{\%\text{ in quantity}}{\%\text{ in price}} = \frac{\text{Percentage change in quantity}}{\text{Percentage change in price}}$$

Agricultural commodities are of low price elasticitices. The price elasticity of demand is low because these commodities are normal necessities and has no good substitutes. They satisfy the human needs. Meanwhile, it is very hard to manufacture complete relative substitute.

The price elasticity of supply is low due high fixed costs associated with production of such commodities, limited flow of market information, and effects of natural hazards.

Low price elasticities lead to unusual fluctuations in farming income. Sometimes, these changes leaf the farmers in worse conditions. In economic history, U.S.A. has protected farmers against these changes through certain agricultural policy instruments.

a.5. Using of Farm and Rural House : There is complete integration between rural house and farm unit. The are one consumption and production unit. Even, sometimes, the resource are used for satisfying the needs of rural family. Note in other sector separation exists between production unit and house. In Egypt, new, the rural women sometimes manages both units.

a.6. Risky Environmental and Financial Difficulties : The agriculture is practiced under open production conditions - Farmers are subject to all hazards. The impact of these hazards on agriculture is relatively strong due to high fixed assets, competitive structure, as well as high price fluctuation.

In addition, due to natural structure of agriculture, there is excessive needs for capital to finance the production and marketing activities. Governments usually face the liquidity problem through facilitating agricultural and rural finance.

B.1.2. Common Approaches to Study Agricultural Economics : Professor El-Attar had pointed two approaches to study agricultural economics problems. They are :

First Approach : Concentrates on endogenous variables related to resource use and economic efficiency. Agricultural economists consider this type of studies under farm management.

Second Approach : Concentrates on all exogenous factors that impact the agricultural industry. These exogenous factors are natural, economic, social, ... etc. For instance the agricultural production is impacted by economic policies, market efficiency and pricing, natural hazards, political considerations, quality of social environments ... etc. Economists usually approaches all of these factors in their studies. But now, in 21st century, the emphases is on the exogenous factors. Since the agriculture is practiced under world market condition.

B.2. Areas of Agricultural Economics Studies : The agricultural economics studies are diversified due to variety of unlimited problems

are in need for solutions. The approaches to solve these problems include qualitative and quantitative approaches. Agricultural economists now days, in 21 st century, use computers and apply mathematical and statistical tools to find solution to agricultural problems. Analysis, interpretation, and predictions, are major operations concerning the resource use and efficiency. The following is are idea about the common areas of agricultural economic studies :

B.2.1. Agricultural Production Economics : This type of studies concentrate on resource use and efficiency to reach a Pareto Optimal situation or in other words to maximize the welfare.

B.2.2. Agricultural Policy : Agricultural policy concentrates on weighing the decisions. Policy objectives and instruments are defined first, then, the policy alternatives are determined. Next, the agricultural policy specialists are weighing these alternatives based on costs benefits or they consider the cost - effective alternative.

B.2.3. Farm Management : This area concentrates on farm as a production unit. Especialists consider the use of available resources to maximize the set farming income or other set of other objectives taking into consideration quality of the availability of resources.

B.2.4. Agricultural Finance : This type of studies concentrate on sources of financing the agricultural activities. Farm credit, credit supply and demand, loan conditions and grantees, liquidity problems, and others concerning the farm finance.

B.2.5. Agricultural Marketing : This area studies markets and marketing definitions, market structure, markets and market economy.

Also, this area include concentration on approaches to study marketing, the marketing functions and margins, market efficiency, area of future markets, and price hedging.

B.2.6. Agricultural Price Analysis : Agricultural prices are subject to all random variations. This because of the nature of agricultural industry. Most of agricultural commodities are of low price elasticities, and subject to seasonality. These facts imply that agricultural prices are characterized by high variations. This area concentrate on agricultural price formulations, future prices and farm income, type and source of price variations, and price expectation models.

B.2.7. Agricultural Cooperation : This area concentrates on the rule of cooperatives in agricultural production. In Egypt, the agricultural cooperatives at all levels are about 5057 cooperatives in 1995. They serve more than 3 million farmers. The role of cooperatives, structure, and cooperative law are subjects of main concern of this area.

B.2.8. Agricultural Planning : Still, even, in market economy, there is needs for indicative planning to over control the resource allocation and use. In Egypt there is still role for ministry of planning and needs for long term planning. The subjects of this area are type and forms of plans, the relation between resources and use, the impact of set plans on welfare ... etc.

B.2.9. Land Economics : This area concentrate on land use, land supply and demand, types of ownership, development of land ownership, physical and economic land classification, and institutional regulations concerning the land use. In following parts, I will concentrate on detailing some these area.

Chapter 2

Agricultural Production Economics

Introduction:

This part is direct translation of out of my book – Prof. Dr. Riad Emarah's book – titled Agricultural Production Economics: Theory and Application" For this section I will introduce the definition of production functions, forms, and implications, Economics of Scale will also be defined, and, then, the three relationships which include input-output, input-input, and out put-output will be detailed.

2.1. Production functions

As stated before, the logic is always mixing the resource use technically and economically to reach the maximum output from a given set of resources. In agriculture, the output is always maximum and it is not possible to increase output from a given resources and technology. The typical definition of a production function is, hence, the maximum output attainable from a set of resources.

Simply we say;

$$Y = f(x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2\text{-}1)$$

Where: Y is output per ton, and x is factor of productive.

This form is deterministic since exact value of Y is associated with each value of x. The probabalistic form of (2-1) is :

$$Y = f(x) + \mu_i \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2\text{-}2)$$

Estimation of this form is possible as long as the distribution of μ_i is known. the compact general form is (2-1) is hard to find in reality. The most realistic form is (x_i^j) resources produces (Y^j) outputs , i.e.,

$$(Y^{if}, Y^{ib}) = f(x_i^j, T) \quad \text{.....................................} \quad (2\text{-}3)$$

This form is the most logical especially if the aim is to discripe agricultural production activities. Variable (T) in (2-3) represents the technology. Form (2-3) had been set by my professor E.O. Heady as:

$$\begin{vmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{vmatrix} = g \begin{vmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{vmatrix} \quad \text{.....................................} \quad (2\text{-}4)$$

2.1.1. Common Forms of Production Function

Economists has utilized many mathematical forms to describe the relations in (2-1)-(2-4). Edward West, Wicksell, and other thought about these definitions handred year ago before writing these notes.

a. One variable cobb-douglas form

This power function had been developed by Charles Cobb and Paul Douglas in 1934. The simplist form is:

$$Y = a X^b \quad \text{.....................................} \quad (2\text{-}5)$$

where a and b are the main parameters. b in this case is the elasticity of production. The function is homogenous of degree (b) and its marginal and Average products are:

$$MPP = \frac{dY}{dX} = baX^{b-1} = (\frac{b}{X})(Y) \quad \text{................} \quad (2\text{-}6)$$

$$APP = \frac{Y}{X} = aX^{b-1} = (\frac{aX^b}{X}) \quad \text{................} \quad (2\text{-}7)$$

From (2-6) and (2-7), we can infer that:

$$MPP = b\,(APP) \quad \text{...} \quad (2\text{-}8)$$

For $b = 1$, the APP = MPP. This form in (2-5) was initially proposed by Wicksell as:

$$Y = a^{\alpha} b^{\beta} c^{\gamma} \qquad \text{...} \quad (2\text{-}9)$$

where: $\alpha + \beta + \gamma = 1$

Also, APP and MPP in (2-6) – (2-7) are homogenous of degree b-1. To proof this fact multiply both sides of (2-6) or (2-7) by factor of proportions $o < \lambda < 1$, then,

$$\begin{aligned} APP &= a (\lambda X)^{b-1} \\ &= a \lambda^{b-1} X^{b-1} = \lambda^{b-1} (aX^{b-1}) \\ &= \lambda^{b-1} (MPP) \end{aligned}$$

b. Two variable Cobb-Douglas form

Most popular case of this general form is

$$Y = A \quad L^{\alpha} K^{\beta} \qquad ; \qquad \alpha + \beta \underset{\leq}{\overset{\geq}{=}} 1 \qquad \text{...................} \quad (2\text{-}10)$$

Where: Y is output, L = Labor, K = Capital, and α, β are parameters. There is a notion behined this form where we can consider factors of production as labor and capital. It is also known to students that K/L= k is the capital / labor ratio. Then, (2-10) can be rewritten as:

$$\frac{Y}{L} = y = A (\frac{K}{L})^{\gamma} \qquad ; \gamma \geq 1 \qquad \text{.....................} \quad (2\text{-}11)$$

Or, productivity per worker is function of a variable capital per worker. As know, in developed economies K/L is high and hence productivity per worker.

The function in (2-10) is homogenous of degree ($\alpha + \beta$), and α is labor share, and β is capital share. If this function exhibit constant returns to scale; i.e., $\alpha + \beta = 1$, then Euler theorm can be applied to distribute output between labor and capital.

APP_L , APP_K, MPP_L, and MPP_K are:

$$\frac{Y}{L} = APP_L = A\ L^{\alpha-1}\ K^{\beta} \qquad \text{.........................} \quad (2\text{-}12)$$

$$\frac{dY}{dL} = MPP_L = A\,\alpha\, L^{\alpha-1}\ K^{\beta} \qquad \text{.........................} \quad (2\text{-}13)$$

$$\frac{Y}{K} = APP_K = A\ L^{\alpha}\ K^{\beta-1} \qquad \text{.........................} \quad (2\text{-}14)$$

$$\frac{dY}{dK} = MPP_K = A\,\beta\, L^{\alpha}\ K^{\beta-1} \qquad \text{.........................} \quad (2\text{-}15)$$

All functions in (2-12) – (2-15) are homogenous of degree $\alpha + \beta - 1$ also, α and β are production elasticities of both labor and capital or:

$$\in_L^p = \frac{MPP_L}{APP_L} = \frac{A\,\alpha\, L^{\alpha-1}\ K^{\beta}}{A\ L^{\alpha-1}\ K^{\beta}} = \alpha \qquad \text{................} \quad (2\text{-}16)$$

$$\in_K^p = \frac{MPP_K}{APP_K} = \frac{A\,\beta\, L^{\alpha}\ K^{\beta-1}}{A\ L^{\alpha}\ K^{\beta-1}} = \beta \qquad \text{................} \quad (2\text{-}17)$$

$\alpha + \beta$ define the economics of scale in this case. Constant returns to scale (CRS) exists if $\alpha + \beta = 1$. Decreasing returns to scale (DRC) exists when the elasticities sum is less than one and increasing returns to scale (IRS) exists visa-versa.

The key question is does any mathematical form serve as a production function?. Alo does any form can descripe any production activity?. The answer on both questions is no. A production function must has a positive first derivative-MPP = $f_i > 0$ – and a negative second derivative. Also, some production activities can be described by a multiplicative form where as the other's such as milk production and fretilizers applications, are not. Then, it is easy to state that not every function can serve as a production function and not any production function is valid to descripe any production processes. The second derivative of (2-12) is:

$$f_{ii} = \frac{d^2Y}{dL^2} = A\,\alpha\,(\alpha - 1)\,L^{\alpha-2}\,K^{\beta} < 0 \quad \text{..................... (2-18)}$$

Where the second cross partial derivative which show the relation between L and K is:

$$\frac{\partial}{\partial K}\left(\frac{\partial Y}{\partial L}\right) = \frac{\partial}{\partial K}(MPP_L) = f_{ij} = A\,\alpha\,\beta\,L^{\alpha-1}\,K^{\beta-1} \geq 0 \ldots \text{ (2-19)}$$

In later parts the value of (f_{ii}) will of great implication in meaning and inplieation.

The production Isoquant, Figure No. (5). The isoequant equation is:

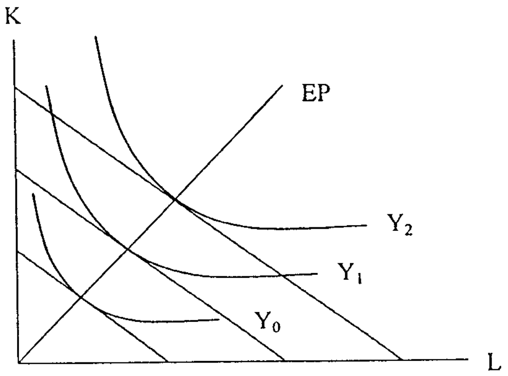


Figure No.(5): Production Iso quants and expansion path.

$$L = \left(\frac{Y}{AK^{\beta}}\right)^{\frac{1}{\alpha}} \quad \text{.. (2-20)}$$

Also, the isoquants in this case has no tangency with any of the two axis.

$$\underset{K\to\infty}{\text{limt}} \frac{dY}{dL} \neq 0 \quad \text{................................. (2-21)}$$

The function is unbounded and hence has no maximum.

The marginal rate of technical substitution ($MRTS_{L,K}$) is:

$$MRTS_{(L,K)} = \frac{dK}{dL} = \frac{MPP_L}{MPP_K} = -\frac{\alpha}{\beta}\left(\frac{K}{L}\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (2\text{-}22)$$

Which implies that increasing both K and L by the same proportion, the substitution will be at constant rate.

The ridge lines for this function are the axses. The production isoclines could be reached by equating the value in (2-22) by negative constant (-M), or

$$(-1)\ MRTS_{(L,K)} = -\ M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}23)$$

or

$$L = M\left(\frac{\alpha}{\beta}K\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}24)$$

$$L = \frac{\alpha\ M}{\beta}(K) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}25)$$

These lines are straight lines and starts from the origin.

c. One Variable Spillman Production Function

Spillman had suggested a production function of form:

$$Y = M - AR^X \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}26)$$

where : Y is total production, X is factor of production, (M-A) is the output level at zero application of (X), and R is the ratio between the marginal products per units i, and i-1.

$$MPP_X = \frac{dY}{dX} = - AR^X \, Log_e^R$$

$$= - AR^X \ln R \qquad \text{.............................. (2-27)}$$

$$APP_X = \frac{Y}{X} = \frac{M - AR^X}{X} = MX^{-1} - AR^{X-1} \qquad \text{........ (2-28)}$$

and

$$\frac{dYi}{dXi} = R = (\frac{dY_{i-1}}{dY_{X_{i-1}}})$$

or $$MP_{X_i} = R \; MPP_{X_i - 1} \qquad \text{.............................. (2-29)}$$

$$\in_P^X = \frac{MPP_X}{APP_X} = \frac{(-AR^X \ln R)X}{(M - AR^X)} \qquad \text{........................ (2-30)}$$

The function is not homogenous of any degree. The elasticity is variable and it could be computed at the average of at any point.

d- Two Variables Spillman Production function

The functional form is:

$$Y = A\,(1 - R_X^X)\;(1 - R_Z) \qquad \text{............................ (2-31)}$$

The production isoquant of the function is:

$$X = Log\left[1 - \frac{Y}{A\,(1 - R^Z)}\right](\log R)^{-1} \qquad \text{...................... (2-32)}$$

And the marginal rate of technical substitution (MRTS) is:

$$MRT_{XZ} = \frac{dX}{dZ} = \frac{(1 - R^{X})\ (R^{Z} \ln R_{Z})}{(1 - R^{Z})\ (R^{X} \ln R_{X})} \quad \ldots\ldots\ldots \quad (2\text{-}33)$$

The Isocline equation could be reahed by equating (2-33) by –K.

These isoclines are not straight lines but passes through the origin. Also, since the function is not homogenous of any degree, the expansion path is not linear. The expansion path equation could be reached by equating (2-33) by the price ratios.

E- Quadratic form

The simplist quadratic form is the most famous production relation. The form is defined as low of dimenishing returns. The common form is:

$$Y = a + b\,X - C\,X^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}34)$$

The main production relations could be infered as:

$$MPP_X = \frac{dY}{dX} = b - 2C\,X > 0 \quad \text{if } X < \frac{b}{2C} \quad \ldots\ldots\ldots\ldots \quad (2\text{-}35)$$

$$\frac{Y}{X} = APP_X = OX^{-1} + b - CX \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}36)$$

$$\in_P^X = \frac{MPP_X}{APP_X} = \frac{b\,X - 2\,C\,X^2}{a + b\,X - C\,X^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}37)$$

and $\in_P^X$ is variable and could be computed to any values of X. The condition for the function to satisfy the requirements as a production function is:

$$\frac{d^2Y}{dX^2} = -2\,C < 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\text{-}38)$$

From (2-35) and (2-38), the function in (2-34) is a production function. Close to the notion of marginal product relations in the case of Spillman production function, equation (2-29), there is a relation between the marginal products of different units as:

$$m_i = m_{i-1} - K$$

or, $$m_i - K = m_{i-1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2\text{-}39)$$

This relation as proved by Heady and Dillon implies that the marginal products are decreasing by an absolute quantities. The function is not homogenous of any degree.

F. Two Variables Quadratic Form.

The quadratic form could be extended to more than one variable, or:

$$Y = a + b_1 X_1 + b_2 X_2 - b_3 X_1^2 - b_4 X_2^2 + b_5 X_1 X_2 \qquad \ldots (2\text{-}40)$$

Student my practice infering the production relations on the same manner indicated before. This function is a production function and of variable elasticities but the function is not homogenous and in cludes interaction part (X_1X_2). This part measures the relation between X_1 and X_2. This relation will be of greate importance later on.

The previous forms of production functions are the most common. The economic literature; however, includes many other forms such as implicit functions, square root, constant elasticity of substitution (CES), . etc. the forms are behind the scope of this course.

2.3. Production Relationships.

The production relationships are defined as:

(1) Input – output.

(2) Input – Input.

(3) Output – output.

2.3.1. Input – Output Relationship

Any one variable production function stated before clearifies this relation. The most common is law of limenishing returns. The simplist form is:

$$Y = F(Xi \mid \overline{X}) \quad \text{.......................................} \quad (2\text{-}41)$$

or, by adding homogenous unit of one variable input, say Lalor, to other fixed imputs output increases first at an increasing rate, and after at a decreasing rate until it reach maximum; then, it decreases for instance, if the production function is of form:

$$Y = 3X + 2X^2 - 0.1X^3 \quad \text{.................................} \quad (2\text{-}42)$$

then, both (MPP_X) and (APP_X) are:

$$MPP_X = 3 + 4X - 0.3X^2 \quad \text{..................................} \quad (2\text{-}43)$$

$$APP_X = 3 + 2X - 0.3X^2 \quad \text{..................................} \quad (2\text{-}44)$$

Then, for different application of X, say 0, 1, 2, ...16, (Y), (MPP_X), and (APP_X) could be calculated. Also, $\in_P^X$ could be estimated. The first stage, $\in_P^X = 1$ ends at using 10 units of X. The second stage; where $0 \leq \in_P^X \leq 1$, will be between using 11-up 14 units of X. then the third stage where $\in_P^X \leq 0$ starts from suing more than 14 units of X. The Economic efficiency occurs only in stage II where $0 \leq \in_P^X \leq 1$, or where:

$$MPP_X = \frac{P_X}{P_Y} \quad \text{..} \quad (2\text{-}45)$$

$$\mathrm{VMP}_X = \mathrm{P}_\mathrm{X} \qquad \text{.. (2-46)}$$

which implies that the economic efficiency occurs when each factor is paid the value of it marginal physical product.

2.3.2. Input–Input Relationship

Two variable production function as:

$$\mathrm{Y} = \mathrm{F}\,(\mathrm{X}_1,\ \mathrm{X}_2) \qquad \text{.. (2-47)}$$

The marginal physical product of X_i is :

$$\mathrm{f}_i = \mathrm{MPP}_{\mathrm{Xi}} = (\frac{\partial \mathrm{Y}}{\partial \mathrm{X}_\mathrm{i}}) = \mathrm{f}\,(\mathrm{X}_1, \mathrm{X}_2) \qquad \text{................ (2-48)}$$

Input–Input relationship examins the changes in (f_1) due to different application of X_2 or simply the impact on one factor on the usefullness of the other factor or:

$$\mathrm{F}_{12} = (\frac{\partial \mathrm{Y}}{\partial \mathrm{X}_1})\ \frac{\partial}{\partial \mathrm{X}_2} \gtreqless 0 \qquad \text{.................................. (2-40)}$$

If the value is zero, the factors are independent since the productivity is not effected by the other factor's application. For positive value of F_{12} the factors are complement, and visa-versa, the factors are substitutes.

Through maximization, the input demand function could be infered or say:

$$\mathrm{X}^*_\mathrm{i} = \mathrm{f}\,(\mathrm{r}_1, \mathrm{r}_2, \mathrm{P}_\mathrm{Y}) \qquad \text{... (2-50)}$$

Where (r_i 's) are prices of inputs and (P_Y) is the price per unit of output. Input demand function in (2-50) could utilized to estimate the value:

$$\frac{\mathrm{dX}_1}{\mathrm{dr}_2} \gtreqless 0 \qquad \text{... (2-51)}$$

This value also uxpremes the relation between the inputs. Hicks and Allen had defined Hicks – Allen elasticity of substitution. It's simplist porm is.

$$\in_c^s = \frac{dX_1}{dr_2} \cdot \frac{r_2}{X_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\text{-}52)$$

The magnitude and value are used to test the relations between inputs.

2.3.3. Output-output Relationship

For a production function of form:

$$(Y_i, Y_j) = f(X_i) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\text{-}53)$$

Student my recall the production possibility fronteer (PPF) defined before in chapter-1. The slope of the curve is the Marginal Rate of Transformation (MRT_r). In agricultural production four major relations are defined by Heady. These four major relations defined by Heady, determined The shape of PPF in each case. The most famous representative relation is the relation between the competitive crops,

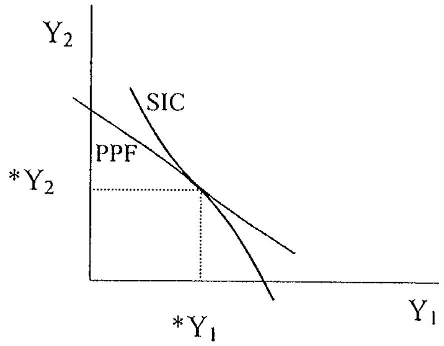


Figure No. (6). Economic efficiency

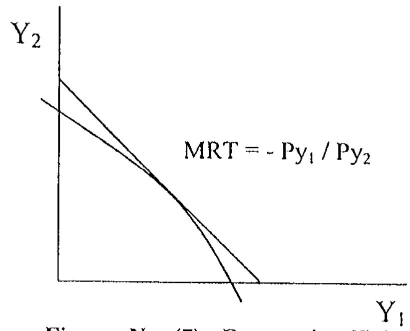


Figure No. (7). Economic efficiency using price ratio.

Figures No. (6) – (7). Other crops start complement or added together in the same production relation and competes afterwards. Even competition among crops takes different forms based on value of MRT_r. For instance, if substitution is constant, there, PPF will be straight line. The substitution in this case in by sacrificing one unit of say (Y_1) to gain one unit of (Y_2).

Chapter 3

Production Under Risk

Decision Making in an Environment of Risk and Uncertainty

This chapter provides a very basic introduction to how risk and uncertainty can be incorporated into farm planning, with an emphasis on the marginal analysis developed in Chapters 2 to 18. Risk and uncertainty are defined. The role of farmer attitudes and objectives in determining particular strategies for dealing with risk and uncertainty is discussed. Expected prices and yields might be used to replace actual prices and yields in marginal analysis models. A simple marginal analysis model incorporating income variability is developed. Alternative strategies for dealing with risk and uncertainty at the farm level are compared.

Key Terms and Definitions:

Risk
Uncertainty
Risk–Uncertainty Continuum
Probability
Expected Income
States of Nature
Action
Consequences
Utility
Utility Function
Variance
Expected Price
Expected Yield
Income Variability
Insurance
Contract
Flexible Facilities and Equipment
Diversification
Government Program

20.1 Risk and Uncertainty Defined

Farmers face situations nearly every day in which the outcomes are uncertain. Nature has a significant impact on farming. For example, it may not rain or it may rain too much. Crops can get hailed out or insects and disease can destroy a crop. An apple or orange crop may get frost, and animals develop diseases and die. Thus farming is inherently linked to the path of nature.

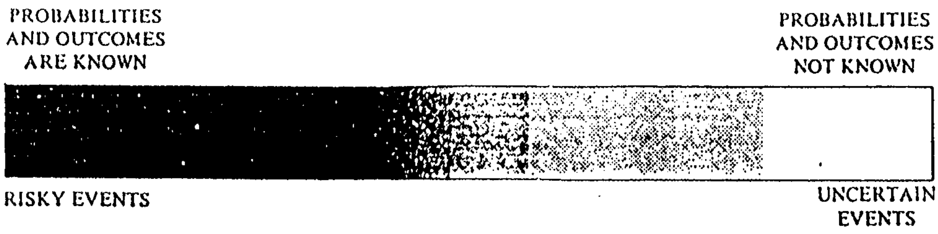


Figure 20.1 Risk–Uncertainty Continuum

The markets affect farmers to a great degree as well. Farmers complain that prices are high when they have nothing to sell and that prices are low when crop yields are high. Prices for agricultural commodities are largely determined by forces outside the control of the individual farmer. Farming takes place in an environment characterized by risk and uncertainty.

Frank Knight was the one initially responsible for making a distinction between the term *risk* and the term *uncertainty*. He argued that in an uncertain environment, possible outcomes and their respective probabilities of occurrence were not known. In a risky environment, both the outcomes and the probabilities of occurrence are known.

Some economists have suggested that to deal with risk, all that is needed is an insurance policy. The insurer can discover the outcomes and the probabilities of their occurrence and write a policy with a premium sufficient to cover the risk and net a profit to the insurer.

Uncertainty cannot be dealt with as easily. If the outcomes and the probabilities associated with each outcome are not known, the insurer would not be able to write a policy with a premium sufficient to cover the risk. Recently, some insurance companies have written policies designed to pay for losses resulting from the occurrence of very unusual events. It is difficult to believe that insurance companies have complete knowledge of the probabilities associated with these events, so distinguishing between risk and uncertainty on the basis of insurability is not the final answer.

Rather than to think of *risk* and *uncertainty* as dichotomous terms, it may be more appropriate to think of a risk–uncertainty continuum (Figure 20.1). At one end of the continuum lie risky events, in which the outcomes and the probabilities attached to each outcome are known. At the other end of the continuum lie uncertain events, in which neither outcomes nor probabilities of their occurrence are known. Many events taking place in farming lie between the polar extremes of risk and uncertainty. Usually, some but not all of the possible outcomes are known, and some but not all outcomes have probabilities attached to them. Much of farming lies midway on the risk–uncertainty continuum.

20.2 Farmer Attitudes Toward Risk and Uncertainty

One of the problems in dealing with risk and uncertainty is that individuals, including farmers, vary markedly in their willingness to take on, and preferences for, risk and uncertainty. No one would normally enter an environment char-

Table 20.1 Alternative Income Generating Strategies

Strategy	Income	Probability
A	$1,000,000	0.3
	−500,000	0.2
	0	0.5
B	100,000	0.3
	50,000	0.4
	0	0.2
	−20,000	0.1
C	50,000	0.7
	30,000	0.2
	0	0.1
D	30,000	0.4
	25,000	0.4
	15,000	0.2

acterized by risk and uncertainty without expectations of gains greater than would be the case in the absence of risk and uncertainty.

That individuals vary markedly in their willingness to take on risk and uncertainty can be illustrated with a simple class game. Suppose that a person is confronted with four different strategies. Each strategy will produce varying levels of income and have probabilities attached to each income level. The four strategies are outlined in Table 20.1. The outcomes and the probability of each outcome are known with certainty. The probability assigned to each strategy represents the expected proportion of times the specified income is expected to occur, relative to the total times the particular strategy (A, B, C, or D) is pursued. For each strategy, the probabilities sum to 1, indicating that for each strategy, only the three income levels are possible. Each member of the class might vote on the strategy that he or she would pursue.

One way to determine which strategy to pursue would be to calculate the expected income occurring as the result of each strategy. The expected income is the income resulting from the strategy weighted by its probability of occurrence. For strategy A, the expected income is $(0.3 \times 1{,}000{,}000) + (0.2 \times -500{,}000) + (0.5 \times 0) =$ \$200,000. For strategy B, the expected income is $(0.3 \times 100{,}000) + (0.4 \times 50{,}000) + (0.2 \times 0) + (0.1 \times -20{,}000) =$ \$48,000. For strategy C, the expected income is $(0.7 \times 50{,}000) + (0.2 \times 30{,}000) + (0.1 \times 0) =$ \$41,000. For strategy D, the expected income is $(0.4 \times 30{,}000) + (0.4 \times 25{,}000) + (0.2 \times 15{,}000) =$ \$25,000. So based on expected income, strategy A would always be pursued, despite the fact that strategy A also allows for the greatest potential losses.

The strategy that is pursued depends in part on the person's particular financial situation. Suppose that if a positive income was not achieved, the person would lack funds necessary to meet the basic needs of life, and would starve. Such a person would be reluctant to pursue any strategy other than D, but a person with $1 million already in the bank would probably choose strategy A. The worst that person could do is lose half of what he or she already had.

The strategy each person chooses is largely unrelated to intelligence or education. There is probably no relationship between the strategy that each

person selects and his or her score on the last hour exam in agricultural production economics. College graduates would not necessarily tend to choose strategies different from high school graduates. All millionaires are not college graduates. Those in bankruptcy are not all high school dropouts.

Each person thus has a different preference for risk and uncertainty versus certainty that is very much intertwined with his or her own psychic makeup. So it is with farmers. Anyone can cite examples of farmers who pursued high-risk strategies that paid off. Examples of farmers who pursued high-risk strategies and went bankrupt are also commonplace, and there are numerous examples of farmers who pursued secure strategies, made a living at farming, but never became wealthy. Self-made millionaires vary widely in intelligence and education, but share a common characteristic in that they are willing to assume large amounts of risk with little, if any, fear if things should not go their way.

Professions vary in the amount of risk. The race car driver assumes enormous amounts of risk in the pursuit of a potentially high payoff. College professors and others in secure, stable occupations are frequently quite risk averse.

Farmers as a group probably prefer to take on more risk than college professors as a group. Nearly every extension agricultural economist has had the opportunity to work with farmers whose incomes exceed the income of the extension agricultural economist several times over. If farmers were not willing to assume some risk, they would have long ago chosen an occupation with a steady income with little variability from year to year. Rather, they let the whims of nature and the marketplace in large measure determine their annual incomes. Students from farm backgrounds sometimes attend an agricultural college in hopes of securing a job that has less income variability than was present on the farm back home.

20.3 Actions, States of Nature, Probabilities, and Consequences

A farmer must have alternatives open in order to make a decision. If two or more alternatives are not available, a decision cannot be made. The alternatives available to a farmer represent the *actions* or *strategies* open to the farmer. The set of actions should encompass the full range of alternatives open to the farmer. In the game in Section 20.2, the actions were represented by the alternative strategies. There are usually a finite number of actions or strategies open to the manager.

The *states of nature* represent the best guess by the decision maker with regard to the possible events that might occur. States of nature are assumed to be outside the control of the decision maker, and in combination with the decision maker's actions determine the outcomes for the decision maker.

Probabilities can be attached to each outcome. They represent the manager's guess as to the number of occurrences of a particular outcome relative to the total number of possible outcomes resulting from a particular strategy. For example, if a particular outcome is expected to occur 3 times out of 10, a probability of 0.3 will be assigned. If all outcomes for each strategy are delineated, the sum of the probabilities associated with each strategy will be 1. This was the case in the game in Section 20.2.

Consequences represent outcomes that are produced by the interaction of the manager's actions and the states of nature. Consequences represent what could happen to the manager. The various income levels represented the outcomes or consequences associated with each strategy in the game.

These terms can be further illustrated with another game. Suppose that the farmer is faced with two options, to grow wheat or soybeans. Assume that nature also has two states, one producing high yields and the other producing low yields. The income resulting from each combination of decision-maker strategies and states of nature, and the corresponding subjective probabilities attached to each state of nature. The resultant matrix is:

	State of Nature and Probabilities	
Action	High Yields: 0.6	Low Yields: 0.4
Grow Soybeans	\$20,000	\$ 3,000
Grow Wheat	\$15,000	\$10,000

The expected income if the farmer grows soybeans is

$$(0.6)(\$20{,}000) + (0.4)(\$3000) = \$13{,}200$$

The expected income if the farmer grows wheat is

$$(0.6)(\$15{,}000) + (0.4)(\$10{,}000) = \$13{,}000$$

If the farmer is interested in maximizing expected income, he or she would be better off to grow soybeans than wheat. However, the farmer might also be concerned with income variability.

20.4 Risk Preference and Utility

The farmer's willingness to take on risk is in large measure linked to his or her psychic makeup. The satisfaction or utility that a farmer receives from each outcome in large measure determines the strategy that he or she will pursue. The maximization of utility subject to constraints imposed by the availability of income is the ultimate goal of a farmer, or for that matter, anyone.

A utility function links utility or satisfaction to the amount of one or more goods that are available. Utility maximization becomes the criterion by which choices are made by the manager. A farmer's utility or satisfaction is not unrelated to his or her expected income, but it is not the same thing as his or her expected income either. If utility and expected income were the same thing, the farmer interested in utility maximization would always choose the strategy that yielded the highest expected income.

In the game outlined in Section 20.2, consider a possible strategy E that yielded \$300,000 with a 0.5 probability, and \$100,000 with a 0.5 probability. If expected income and utility were the same, everyone would be indifferent between this strategy and strategy A presented in Table 20.1. Most people prob-

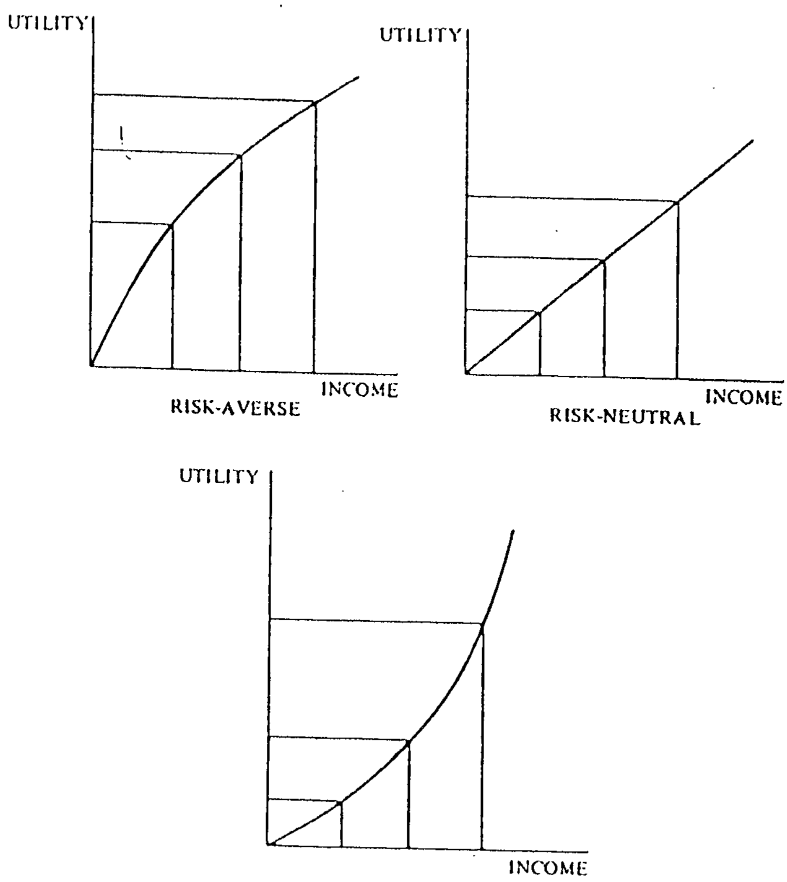


Figure 20.2 Three Possible Functions Linking Utility to Income

ably would strongly prefer strategy E to strategy A, despite the fact that both strategies yield the same expected income of $200,000. Clearly, there is more to maximizing utility than maximizing expected income.

A good deal of effort by economists has been devoted to proofs that utility functions for individuals and, in particular, for farm managers exist. Figure 20.2 illustrates three possibilities with respect to possible functions linking utility to income. Assuming that the farmer can achieve greater income only at the expense of taking on greater risk or uncertainty, the risk averter will have a utility function that increases at a decreasing rate as income rises. The utility function for the risk-neutral person will have a constant slope. The utility function for the risk preferrer will increase at an increasing rate.

One utility function that is sometimes assumed is the quadratic utility function

$$U = z + bz^2 \tag{20.1}$$

where z is some variable of concern that generates utility for the manager, such as income. Suppose that there exists uncertainty with regard to the income

level, so that z is replaced by the an expected z or $E(z)$. Therefore, expected utility is

$$E(U) = E(z) + bE(z^2) \tag{20.2}$$

The expected value of a squared variable is equal to the variance of the variable plus the square of the expected value. Therefore,

$$E(z^2) = \sigma^2 + [E(x)]^2 \tag{20.3}$$

Hence

$$E(U) = E(x) + b[E(x)]^2 + b\sigma^2 \tag{20.4}$$

Thus utility is a function not only of expected income, but also its variance.

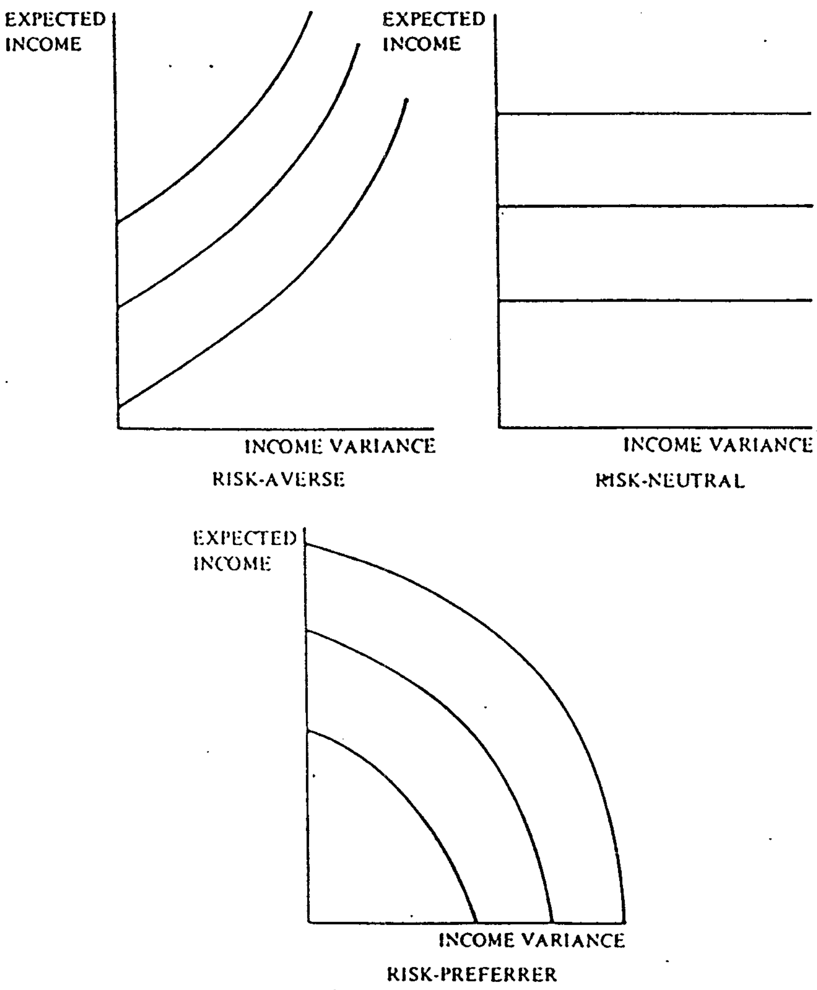


Figure 20.3 Indifference Curves Linking the Variance of Income with the Expected Income

Indifference curves that show possible combinations of income and its variance that yield the same amount of utility for the manager might be obtained by assuming that U equals $U°$ and taking the total differential of the utility function:

$$dU° = 0 = (1 + 2b)dE(x) + bd(\sigma^2) \quad (20.5)$$

Therefore,

$$dE/d\sigma^2 = -b/[1 + 2bE(x)] \quad (20.6)$$

The denominator $[1 + 2bE(x)]$ will always be positive. The shape of the indifference curves will depend on the value of b. If b is zero, the farmer neither desires nor dislikes risk. The farmer is risk neutral. If b is positive, the farmer loves risk, and indifference curves will have a negative slope. If b is negative, the farmer is risk averse and will have indifference curves sloping upward to the right. Figure 20.3 illustrates some possible relationships suggested by this utility function.

20.5 Risk, Uncertainty, and Marginal Analysis

The models in this text that used marginal analysis all assumed that input prices, output prices, and outputs were known with certainty. There exist several ways of incorporating risk and uncertainty into these models, while relying on marginal analysis as the basic tool for decision-making information.

One of the simplest ways of incorporating risk and uncertainty into a model might be to use expected prices or yields rather than actual prices or yields within the model. Just how yield and price expectations are formulated by farmers has been a topic of great concern to some agricultural economists.

An agricultural economist interested in the futures market might argue that one way a farmer formulates price expectations is by studying the prices on futures contracts for the month in which the crops or livestock are expected to be marketed. The futures market does not necessarily predict with a high degree of accuracy what the cash price will be some time in the future. However, the prices for futures contracts are an additional piece of information that the farmer might be able to use at least as a partial basis for developing expectations with regard to prices at marketing time. The farmer might also take advantage of the futures market to determine specific prices at the time of delivery, and these prices could be treated the same as a certain price within the model.

Farmers have many other sources of information with regard to expected prices. The news media, farm magazines, the agricultural extension service, the federal government, and private price forecasting agencies all devote considerable effort to providing price and general outlook information for farmers. One problem with this information is that the quality can vary widely. The farmer must not only study the forecasts obtained from each source, but also attach subjective probabilities with respect to its accuracy.

Farmers rely heavily on current and recent past prices as a means of formulating price expectations. If the cash price of corn at the start of the production season is high relative to soybeans, almost certainly there will be an increase in corn acreages irrespective of what prices are forecasted to prevail

at the time the crop is marketed. Current and recent past cash prices may not accurately represent the prices that should be included in a profit-maximization model.

Yield or output expectations are usually largely based on past experience with the particular commodity. Suppose that a farmer experienced corn yields of 130 bushels per acre last year, 114 bushels per acre the year before, and 122 bushels per acre the year before that. A simple way of formulating a yield expectation might be to average the yield over the past three years. This would treat each of the past three years as equally important in the formulation of the yield expectation. In this example, the expected yield would be 122 bushels per acre.

Another way would be to weight more heavily data from the recent past relative to earlier data. Expected output becomes a distributed lag of past output levels. For example, a farmer might place a weight of 0.6 on last year's data, 0.3 on the year before, but 0.1 on the year previous to that. The weights representing the relative importance of each year's data are highly subjective but should sum to 1. The expected yield in this example is

$$y = 0.6(130) + 0.3(114) + 0.1(122) = 124.4 \text{ bushels per acre.} \quad (20.7)$$

Once price and output expectations have been formulated, they could be inserted directly into the model. The marginal conditions would then be interpreted based on expected rather than actual prices.

The major disadvantage of using expected price and output levels as the basis for formulating economic models is that the approach fails to recognize that price and output variability leads to income variability for the farmer. Only if the farmer is risk neutral is the expected profit maximum optimal for the farmer. Despite the fact that a model using expected prices and output levels leads to maximum profits when expected prices and outputs are realized, income variability when expected prices and yields are not realized may lead to severe financial problems for the farmer. Even if expected prices and outputs are accurate over a planning horizon of several years, the farmer must survive the short-run variability in order to make the long run relevant.

One way of incorporating such variability into a model would be to add additional constraints. Suppose that the farmer used an input bundle to produce two outputs, y_1 and y_2. Due to price and output instability, there is income variability associated with both y_1 and y_2. The income variability associated with y_1 is $y_1\sigma_1^2$, and the income variability associated with y_2 is $y_2\sigma_2^2$. The income variability associated with the first commodity may partially offset or add to the income variability from the second commodity. An interaction term or covariance term is needed. This term that adjusts for income variability interaction is $2y_1y_2\sigma_{12}$.

The total income variability (δ) is

$$\delta = y_1\sigma_1^2 + y_2\sigma_2^2 + 2y_1y_2\sigma_{12} \quad (20.8)$$

The farmer is interested in maximizing revenue subject to the constraint that income variability not exceed a specified level $\delta°$, and the constraint imposed

by the availability of dollars for the purchase of the input bundle x. So the Lagrangian is

$$L = p_1 y_1 + p_2 y_2 + \psi(\delta^\circ - y_1\sigma_1^2 - y_2\sigma_2^2 - 2y_1y_2\sigma_{12}) + \eta[vx^\circ - vg(y_1, y_2)] \tag{20.9}$$

The corresponding first-order conditions are

$$\partial L/\partial y_1 = p_1 - \psi(\sigma_1^2 + 2y_2\sigma_{12}) - \eta v g_1 = 0 \tag{20.10}$$

$$\partial L/\partial y_2 = p_2 - \psi(\sigma_2^2 + 2y_1\sigma_{12}) - \eta v g_2 = 0 \tag{20.11}$$

$$\partial L/\partial \psi = \delta^\circ - y_1\sigma_1^2 - y_2\sigma_2^2 - 2y_1y_2\sigma_{12} = 0 \tag{20.12}$$

$$\partial L/\partial \eta = vx^\circ - vg(y_1, y_2) = 0 \tag{20.13}$$

If there were no income variability, the first-order conditions would be the same as the standard first-order conditions in the product-product model. Income variability could reduce or increase the output of y_1 relative to y_2. The signs on the income variance–covariance terms are indeterminate. Income variability can be incorporated into a standard model, but the key problem with this is that the farmer would need to be able to provide an indication of the variances and covariances associated with the incomes obtained from the commodities being produced.

20.6 Strategies for Dealing with Risk and Uncertainty

A farmer has a number of strategies available for ameliorating the impacts of risk and uncertainty. Each of these strategies reduces losses when nature is unfavorable or the markets turn against the farmer, but also reduce potential profits when nature and the markets are favorable.

20.6.1 Insure against risk

If an insurance policy is available, income variability due to that source of risk can be reduced by purchasing the policy. People purchase fire insurance not because they expect their house to burn down, but because the cost of the insurance is low relative to the potential loss that could occur should the house burn. Insurance policies work best when the probability attached to the occurrence of the event is low, but if the event occurs, the result would be catastrophic. In other words, insurance should be used in situations where there is a low probability of a large loss.

Crop insurance plans have the effect of making the farmer's income from one year to the next more even, despite the fact that the farmer may pay in the form of premiums somewhat more than is returned in the form of claims over a 10-year period. The premium cost reduces potential profits in years without a crop loss. Only if the risk of crop failure on a particular farm substantially

exceeds the risks on which the premiums were based will returns from the insurance policy more than offset premium costs.

20.6.2 Contracts

The futures market can be thought of as a device which allows farmers to contract for the sale of a specified commodity at a specified price for delivery at some future point in time. Thus the futures market is a mechanism to reduce or eliminate price uncertainty by determining prices to be paid after harvest, or at the point when the commodity is ready for market. Although price and income variability will be reduced, in a rising market, the farmer will limit potential gains if prices are determined at the start of the production season.

The futures market is but one contractual arrangement for eliminating price uncertainty. Any contractual arrangement that at the start of the production season specifies a price to be received at the end of the production season will eliminate price uncertainty. Contractual arrangements are commonly used for commodities such as broilers, horticultural crops, and sunflowers. Any contractual price would work well in a marginal analysis model, since it represents price certainty.

20.6.3 Flexible facilities and equipment

If a farmer is to adjust to changing relative product and input prices, it must be possible to adapt buildings and equipment lasting more than one production season to alternative uses as input and output price ratios change. Figure 20.4 illustrates some possibilities. The long-run product transformation function represents the possibilities open to the farmer before buildings are built and equipment is purchased (curve A). Once the durable items have been built and capital committed, two possibilities exist.

Specialized facilities will allow production to take place on the long-run planning curve if relative price ratios turn out to be as expected over the long

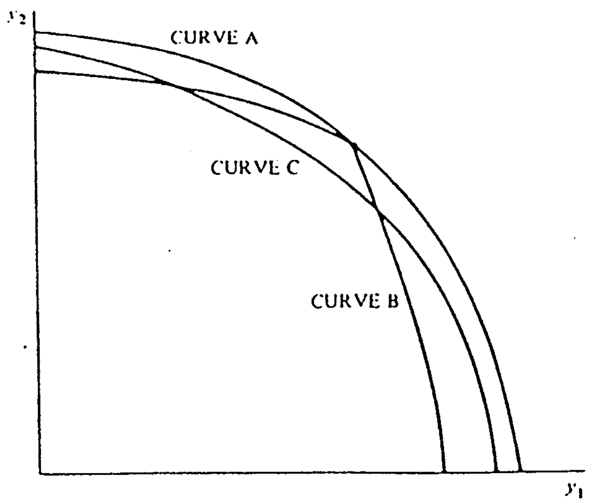


Figure 20.4 Long-Run Planning: Specialized and Nonspecialized Facilities

term. But production drops off dramatically if the use of the buildings and equipment is changed to produce another mix of outputs in response to changing relative prices (curve B). A milking parlor is an example of a livestock facility ill adapted to other uses. Specialized harvesting equipment for a new crop not previously grown by the farmer (such as sunflowers in a farming area devoted to wheat and other small grains) is another example.

If a good deal of price variability is expected, a better strategy might be to construct buildings and to purchase machinery and equipment adapted to a wide variety of uses with little additional cost (curve C). A point on the long-run planning curve is never achieved under any conceivable output price ratio. A barn suited for the production of many different classes of livestock is an example of a flexible facility. In grain production, planting tillage and harvest equipment adapted to an array of different crops represents flexible equipment. The farmer is better off under extreme price variability with flexible facilities and equipment. The farmer is better off with the specialized facilities if price variability is not extreme.

A farmer who attempts to deal with price uncertainty by choosing to build or purchase machinery and facilities adaptable to a diverse array of uses is, in effect, choosing facilities allowing for a greater elasticity of substitution on the product side. A facility suitable only for the production of one commodity, or two commodities in an exact fixed proportion to each other, would lead to a zero elasticity of substitution on the product side.

20.6.4 Diversification

Diversification is a strategy long used by farmers for dealing with both price and output uncertainty. The idea behind a diversification strategy is to let profits from one type of livestock or crop enterprise more than offset losses in another enterprise. Diversification may also make more effective use of labor and other inputs throughout the year, thus increasing income in both good years and bad. To deal most effectively with price and income variability, the enterprises on the diversified farm must have prices and outputs that move opposite to each other.

It does little good to attempt to reduce output variability by both growing wheat and raising beef cattle, if wheat yields are low when rainfall is inadequate, and at the same time, beef cattle cannot be adequately fed on pastures with inadequate rainfall. To guard against uncertainty associated with drought, the farmer would need to find an enterprise in which the output is not as rainfall dependent, and this may be difficult.

The strategy may be more effective for dealing with price uncertainty. Beef and grain prices sometimes move together, but not always, nor do beef and pork prices always move in tandem. The ideal strategy would involve locating commodities whose prices always move in opposite directions. While a farmer who diversifies may substantially reduce income variability and make more effective use of certain inputs, income could also be reduced relative to what would have occurred had production of only the high-priced commodity taken place. The diversified farmer also bears a cost in not as effectively being able to take advantage of pecuniary and other internal economies open to the specialized counterpart.

20.6.5 Government programs

The federal government long has been heavily involved in programs that provide price and income support for farmers. Agricultural policy during the 1970s moved away from mandatory programs and toward programs that allow the farmer to decide for himself or herself whether or not to participate. Most government programs have been directed toward the reduction of price, rather than output uncertainty, but the wheat and feed grain disaster programs of the 1970s are examples of programs designed to support farm incomes when output levels are low.

Net farm income for the United States is rather unrelated to output in a particular year. The 1983 drought throughout much of the midwestern grain-producing areas dramatically reduced output of key crops such as corn, although net farm income was higher in 1983 than in 1982, when drought was not widespread but prices were lower. A farmer's income increases when success is achieved at growing a crop in which other farmers had widespread failures.

Government price support programs that place floors under which commodity prices are supported are usually thinly disguised mechanisms for supporting farm incomes. Such programs increase incomes and support the welfare of every farmer who participates, large and small. Participation in a program will normally reduce income variability, and to the extent that tax revenues for supporting prices come from nonfarm consumers, long-term income may also be larger than would have been the case if the program had not been in place.

When given a choice, occasionally farmers will find it to their advantage not to participate in a government program. The decision can be made by firs calculating net revenue when the farmer participates. This usually means a restricted output (y) at a high price. Net revenue based on nonparticipation is then calculated assuming more output but a lower price. However, the decision by the farmer to participate or not participate will be based both on the extent to which participation in the program will reduce income variability as well as increase net income.

Recently, the federal government has been making attempts to move away from federal price support programs. For programs that remain, increasingly farmers are being asked to pay for the full cost of government price support programs, including the cost of storage of commodities in excess supply. The recent move toward a no net cost tobacco program could be an indication of potential programs for other commodities.

When government support prices exceed levels at which supply and demand are in equilibrium, surpluses of the price-supported commodities occur. Most commodities cannot be stored indefinitely, and storage costs can quickly become rather high. In the past, the government has used the school lunch program to dispose of surplus, government-owned commodities. Recently the government has distributed surplus dairy products occurring as a result of the price support program to low income and elderly residents. Unfortunately, the federal government does not have the option of giving away cigarettes to low-income or elderly people, or making chewing tobacco an approved vegetable on the school lunch menu.

In the past, government programs have both reduced income variability and raised net farm incomes. Utility is increased because incomes rise and variability in incomes is reduced. A no-net-cost program would only reduce income variability. Therefore, a no-net-cost government program could increase utility if farmers were not risk neutral. However, incomes to farmers (and utility) over the long term would be reduced because of the cost to farmers of operating the government program.

20.7 Concluding Comments

This chapter has provided a very basic introduction to the problem of taking into account risk and uncertainty in economic analysis. Specific models incorporating risk and uncertainty could easily fill an entire textbook. The simplest approaches for including risk and uncertainty involve replacing actual prices and yields with the respective expected values. However, price and income variability leads to income variability, which in turn, affects the farmer's utility or satisfaction.

While marginal analysis can form the basis for some models that include risk and uncertainty, other models are based on approaches that do not require the traditional framework. Included in the latter category are approaches involving games such as those outlined in Section 20.3. The reading list at the end of this text includes a number of articles dealing with risk and uncertainty using a variety of modeling approaches.

Problems and Exercises

1. Calculate the expected income on the basis of the following data:

Income	Probability
$100,000	0.2
20,000	0.5
−50,000	0.3

2. Why are expected income and utility not the same thing?
3. Why do farmers not always choose to pursue the strategy with the greatest expected income?
4. Discuss possible states that nature might assume in farming, and possible actions a farmer might take in dealing with these states of nature.
5. Suppose that an enterprise with a greater expected income also resulted in a greater input variability than that for another enterprise. How could this situation be considered within a marginal analysis framework?
6. Suggest strategies that a farmer might use to deal with risk and uncertainty.

Reference

Knight, Frank H. *Risk, Uncertainty and Profit.* Boston: Houghton Mifflin, 1921.

Appendix

Optimality (Numerical Example)

function—is known, some recommendations about input use can be made even though prices are not specified.

First, if the product has any value at all, input use, once begun, should be continued until Stage II is reached. That is because the efficiency of the variable resource, measured by *APP*, increases throughout Stage I; it is not reasonable to cease using an input when its efficiency in use is increasing. For the production function in Figure 2-4, at least 10 units of input should be used.

Second, even if the input is free, it will not be used in Stage III. The maximum total output occurs on the upper boundary of Stage II; further input increments decrease output. It is not reasonable to increase input use when total product is decreasing. Thus, in Figure 2-4, the largest amount of variable input that would be used is 14 units.

Stage II and its boundaries are the area of economic relevance. Variable input use must be somewhere in Stage II, but the exact amount of input can be determined only when choice indicators, such as input and output prices, are known. If production is undertaken at all, then the objective will be achieved somewhere in Stage II.

Stages Of Production—Algebraic Interpretation

The interpretation of the three stages of production and their delineation on the basis of the relationships between the *APP* and *MPP* can be deduced from data of Table 2-2 and Figure 2-4. The same conclusions can be reached using algebraic calculations.

The slope of *TPP* is zero when *TPP* reaches its maximum. Since the *MPP* equation defines the slope at any level of input, *X*, the amount of *X* at which *TPP* reaches its maximum can be obtained by equating the *MPP* equation to zero:

$$MPP = 3 + 4X - 0.3X^2 = 0, \quad \text{from which}$$

$$X = \frac{-4 - \sqrt{16 + 3.6}}{-0.6} = 14.04 \qquad (2.2)$$

Thus, when $X = 14.04$, *TPP* reaches its maximum and *MPP* equals zero. This gives the boundary between Stages II and III. It locates the point where the tangent to the production function has a zero slope.

Similarly, the first derivative of the *APP* equation defines the slope at any level of input, *X*, on the *APP* curve. When *APP* is at its maximum, the slope of the *APP* equals zero. From the equation (2.1)—the classical production function,

$$APP = 3 + 2X - 0.1X^2 \qquad (2.3)$$

$$\frac{dAPP}{dX} = 2 - 0.2X = 0 \quad \text{from which} \quad X = \frac{2}{0.2} = 10 \qquad (2.4)$$

APP reaches its maximum when $X = 10$. At that point *APP* also

equals *MPP*. By substituting $X = 10$ into the *MPP* and *APP* equation, it can be shown that $MPP = 13$ and $APP = 13$. At this point, the average and the marginal rate at which input, X, is transformed into product, Y, is equal and a line through the origin is tangent to the production function. This result can be derived by differentiating the general expression for *APP*; that is, by differentiation of $f(X)/X$ with respect to X.

Elasticity Of Production And Point Of Diminishing Returns

A discussion of the law of diminishing returns and the classical production function inevitably leads into the determination of the "point" of diminishing returns, meaning the input and yield amount at which returns begin to diminish. But, what is that point? The law itself is ambiguous. Study of Figure 2-4 shows that marginal physical product begins to decrease at an input level of 6.7, the point of inflection on the production function, where *MPP* is at the maximum. Average physical product begins to decrease at 10 units of input, and total physical product begins to decrease at 14 units. Clearly, the point of diminishing returns depends on which of these three measures is being discussed.

To avoid this, some writers apply the law of diminishing returns directly to the marginal product. That is, they call it the law of diminishing marginal returns and specify in the definition that, as successive units of the variable input are added, marginal returns will eventually decrease. It is appropriate to define the law of diminishing returns in terms of the marginal product. Some ambiguity is caused, however, because the point of diminishing marginal returns, occurring at an input of 6.7 in Figure 2-4 differs from the boundary of Stage II, occurring where X is 10. A solution using the elasticity of production has been suggested by Cassels.[2]

The elasticity of production is a concept that measures the degree of responsiveness between output and input. The elasticity of production, like any other elasticity, is independent of units of measure. Elasticity of production (ϵ_p), is defined as

$$\epsilon_p = \frac{\text{Percent change in output}}{\text{Percent change in input}}$$

From this, the elasticity of production is determined to be

$$\epsilon_p = \Delta Y/Y \div \Delta X/X = \frac{X}{Y} \cdot \frac{\Delta Y}{\Delta X} = \frac{MPP}{APP}$$

In Stage I *MPP* is greater than *APP* (see last column in Table 2-2). Therefore, ϵ_p is greater than one. In Stage II *MPP* is less than *APP* and ϵ_p is less than one but greater than zero. In Stage III *MPP* is negative and ϵ_p is negative.

The elasticity of production figures in Table 2-2 represent exact elasticities as opposed to average or "arc" elasticities that can be

calculated by dividing average MPP by APP. The exact ϵ_p is derived by dividing the exact marginal physical product by the average physical product at any level of X.

$$\epsilon_p = \frac{dY}{dX} \cdot \frac{X}{Y}$$

From the classical production function (2.1)

$$MPP = \frac{dY}{dX} = 3 + 4X - 0.3X^2$$

and

$$\frac{1}{APP} = \frac{X}{Y} = \frac{X}{3X + 2X^2 - 0.1X^3} = \frac{1}{3 + 2X - 0.1X^2}$$

and

$$\epsilon_p = \frac{3 + 4X - 0.3X^2}{3 + 2X - 0.1X^2} \qquad (2.5)$$

When $X = 1$ then $\epsilon_p = 6.7/4.9$ or 1.37; when $X = 10$ then $\epsilon_p = 13/13$ or 1.0, as shown in Table 2-2. If the elasticity of production is equal to one, a 1 percent change in input will produce a 1 percent change in output. If ϵ_p is greater (less than) one, then a 1 percent change in input will bring about a greater than (less than) 1 percent change in output.

The "point" of diminishing returns can be defined to occur where $MPP = APP$ and ϵ_p is one—the lower boundary of Stage II. This is the minimum amount of variable input that would be used, and it occurs where the efficiency of the variable input is at a maximum. Using this definition, it can be argued without knowing input or output prices that input use will always be extended to the point of diminishing returns. At the other boundary of Stage II, MPP equals zero and hence ϵ_p also equals zero. Thus the relevant production interval for a variable input is that interval wherein $0 \leqq \epsilon_p \leqq 1$.

COSTS OF PRODUCTION

Costs are the expenses incurred in organizing and carrying out the production process. They include outlays of funds for inputs and services used in production. In the short run total costs include fixed and variable costs. In the long run all costs are considered variable costs because all inputs are variable.

Fixed And Variable Costs

A resource or input is called a fixed resource if its quantity is not varied during the production period. A resource is a variable

resource if its quantity is varied at the start of or during the production period. Most inputs have costs associated with them. In general, costs of fixed inputs are called fixed costs, while costs of variable inputs are called variable costs.

Fixed costs do not change in magnitude as the amount of output of the production process changes and are incurred even when production is not undertaken. Fixed costs are independent of output. In farming, cash fixed costs include land taxes, principal and interest on land payments, insurance premiums, and similar costs. Noncash fixed costs include building depreciation, machinery and equipment depreciation caused by the passing of time, interest on capital investment, charges for family labor, and charges for management.

Fixed costs are usually associated with fixed inputs (technical units) because when the amount of a resource is fixed, the costs associated with it are also fixed. Some care must be taken with this definition because it may not always be true. A fixed cost requires only that the cost to the farmer be constant over the production period; not that the amount of input used be constant. For example, consider the farmer who subscribes to a rural electric utility and agrees to pay a monthly charge regardless of the amount of electric power consumed. In this case, the cost of electricity is fixed, but the amount used can be varied.

Computation Of Total Costs

Table 2-3 includes the computation of costs for the classical production function presented in Figure 2-2 and Table 2-2. The production function is presented in columns 1 and 2 of Table 2-3. The cost per unit of variable input is assumed to be $100. Fixed costs are assumed to be $1,000. For simplicity, the $1,000 is assumed to represent exactly the costs associated with the fixed inputs used in the production process, and the only variable cost is that incurred in purchasing the variable input.

Total fixed costs, *TFC*, are shown in the third column of Table 2-3. These costs are the same for all output levels. Thus, once computed from the farm or enterprise budget, *TFC* are known and unchanging. *TFC* are shown graphed in Figure 2-5. *TFC* are $1,000 for all output levels; this is represented by a straight line parallel to the horizontal or *Y* axis and located 1000 units up the vertical scale.

Total variable cost, *TVC*, is computed by multiplying the amount of variable input used by the price per unit of input. In symbolic notation, if X is the amount of variable input used and P_x is the price or cost per unit of input,[3] then

$$TVC = P_x X$$

From Table 2-3, when 4 units of input are used, $TVC = \$100 \cdot 4 = \$400$. When 12 units of input are used, $TVC = \$100 \cdot 12 = \$1{,}200$, etc. A graph of *TVC*, computed in column 4 of Table 2-3, is shown in Figure 2-5. *TVC* is zero when output, and consequently

TABLE 2-3

PRODUCTION COSTS DERIVED FROM THE CLASSICAL PRODUCTION FUNCTION (P_x = $100)

(1) X	(2) Y	(3) TFC	(4) TVC	(5) TC	(6) AFC	(7) AVC	(8) ATC	(9) MC Average	(10) MC Exact
0	0	$1,000	$ —	$1,000	$ —	$ —	$ —	$ —	$ —
1	4.9	1,000	100	1,100	204.1	20.1	224.1	20.4	14.9
2	13.2	1,000	200	1,200	75.8	15.2	91.0	12.0	10.2
3	24.3	1,000	300	1,300	41.1	12.3	53.4	9.0	8.1
4	37.6	1,000	400	1,400	26.6	10.6	37.2	7.5	7.0
5	52.5	1,000	500	1,500	19.0	9.5	28.5	6.7	6.5
6	68.4	1,000	600	1,600	14.6	8.8	23.4	6.3	6.2
7	87.4	1,000	700	1,700	11.4	8.0	19.4	6.1	6.1
8	100.8	1,000	800	1,800	9.9	7.9	17.8	6.2	6.3
9	116.1	1,000	900	1,900	8.6	7.8	16.4	6.5	6.8
10	130.0	1,000	1,000	2,000	7.7	7.7	15.4	7.2	7.7
11	141.9	1,000	1,100	2,100	7.0	7.8	14.8	8.4	9.3
12	151.2	1,000	1,200	2,200	6.6	7.9	14.5	10.8	12.8
13	157.3	1,000	1,300	2,300	6.4	8.3	14.7	16.4	23.3
14	159.6	1,000	1,400	2,400	6.3	8.8	15.1	43.5	5,000.0
15	157.5	1,000	1,500	2,500	6.4	9.5	15.9	—	—
16	150.4	1,000	1,600	2,600	6.7	10.6	17.3	—	—

the input, is zero. It increases as output increases. The shape of the *TVC* curve depends upon the shape of the production function; for the classical production function, *TVC* is always shaped as in Figure 2-5.

Total costs, *TC*, are the sum of total variable cost and total fixed costs. They are presented in column 5 of Table 2-3 and obtained by adding *TVC* and *TFC* for any output level. For an output of 151.2 units, *TC* are $1,000 plus $1,200 or $2,200. *TC* for other output levels are obtained similarly. When no variable input is used, *TC* = *TFC*. Total costs are graphed in Figure 2-5. The *TC* curve is equal to the vertical addition of *TFC* and *TVC*. It is shaped exactly like the *TVC* curve but in this example, it is always 1000 units higher on the vertical axis. The shape of the *TC* curve, like that of the *TVC* curve, depends upon the production function. In symbolic notation, *TC* can be written

$$TC = TFC + TVC = TFC + P_x X$$

Average Fixed Costs, Average Variable Cost, Average Total Costs

Average fixed costs, *AFC*, are computed by dividing total fixed costs by the amount of output. *AFC* varies for each level of output; as output increases, *AFC* decreases. Thus, when economists refer to increasing output as a method of "spreading fixed costs," they mean increasing production to divide total costs among an increased number of units of output, thereby reducing costs per unit. *AFC* are presented

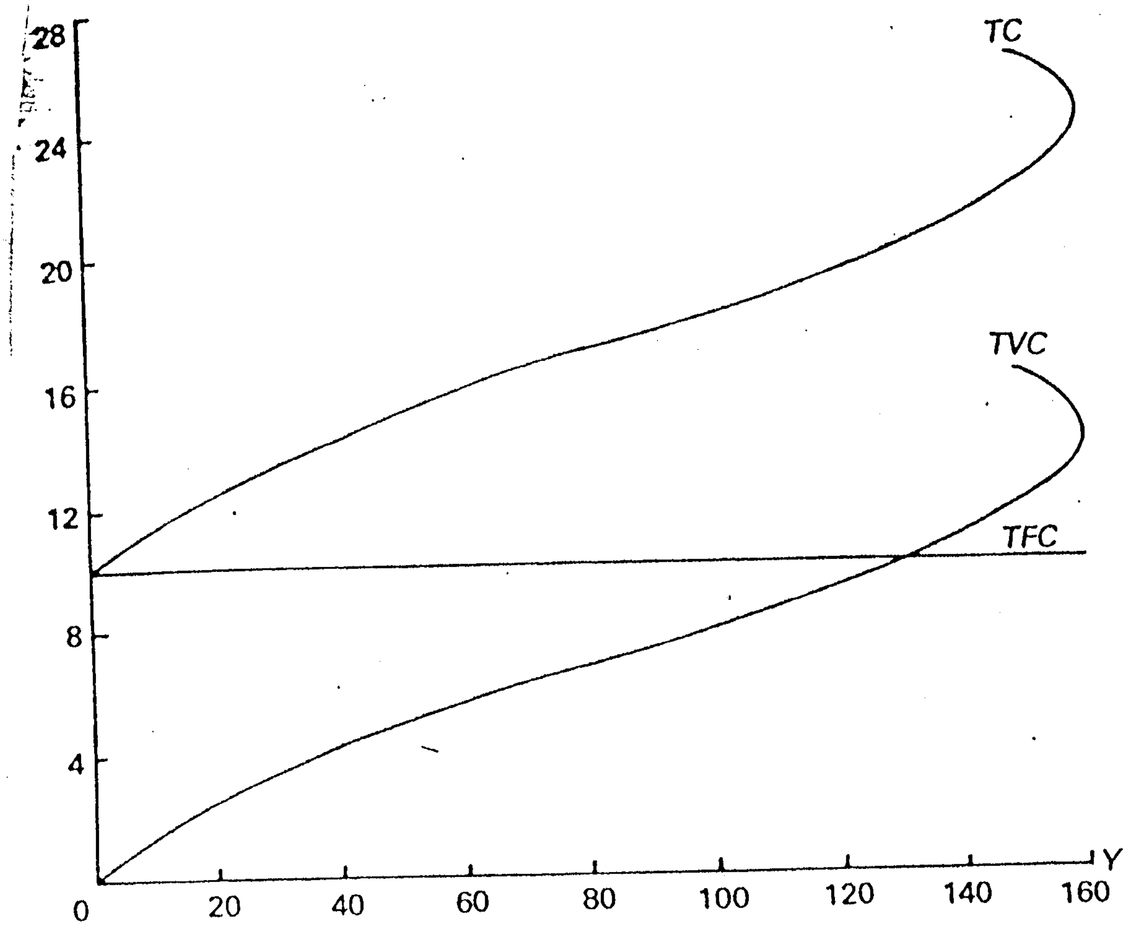


FIGURE 2–5
COST CURVES FOR THE CLASSICAL PRODUCTION FUNCTION

in column 6 of Table 2-3 and graphed in Figure 2-6. *AFC* for output amounts of 24.3 and 141.9 are:

$$Y = 24.3: \quad AFC = \frac{TFC}{Y} = \frac{\$1{,}000}{24.3} = \$41.1$$

$$Y = 141.9: \quad AFC = \frac{TFC}{Y} = \frac{\$1{,}000}{141.9} = \$7.0$$

When output is zero, *AFC* cannot be computed because division by zero is not permissible. *AFC* always has the same shape regardless of the production function. Its general location on the graph depends upon the magnitude of total fixed costs.

Average variable cost, *AVC*, is computed by dividing total variable cost by the amount of output. *AVC* varies depending on the amount of production; the shape of the *AVC* curve depends upon the shape of the production function. The height of the *AVC* curve depends upon the unit cost of the variable input. *AVC* is computed in column 7 of Table 2-3 and graphed in Figure 2-6. *AVC* for two different output amounts is

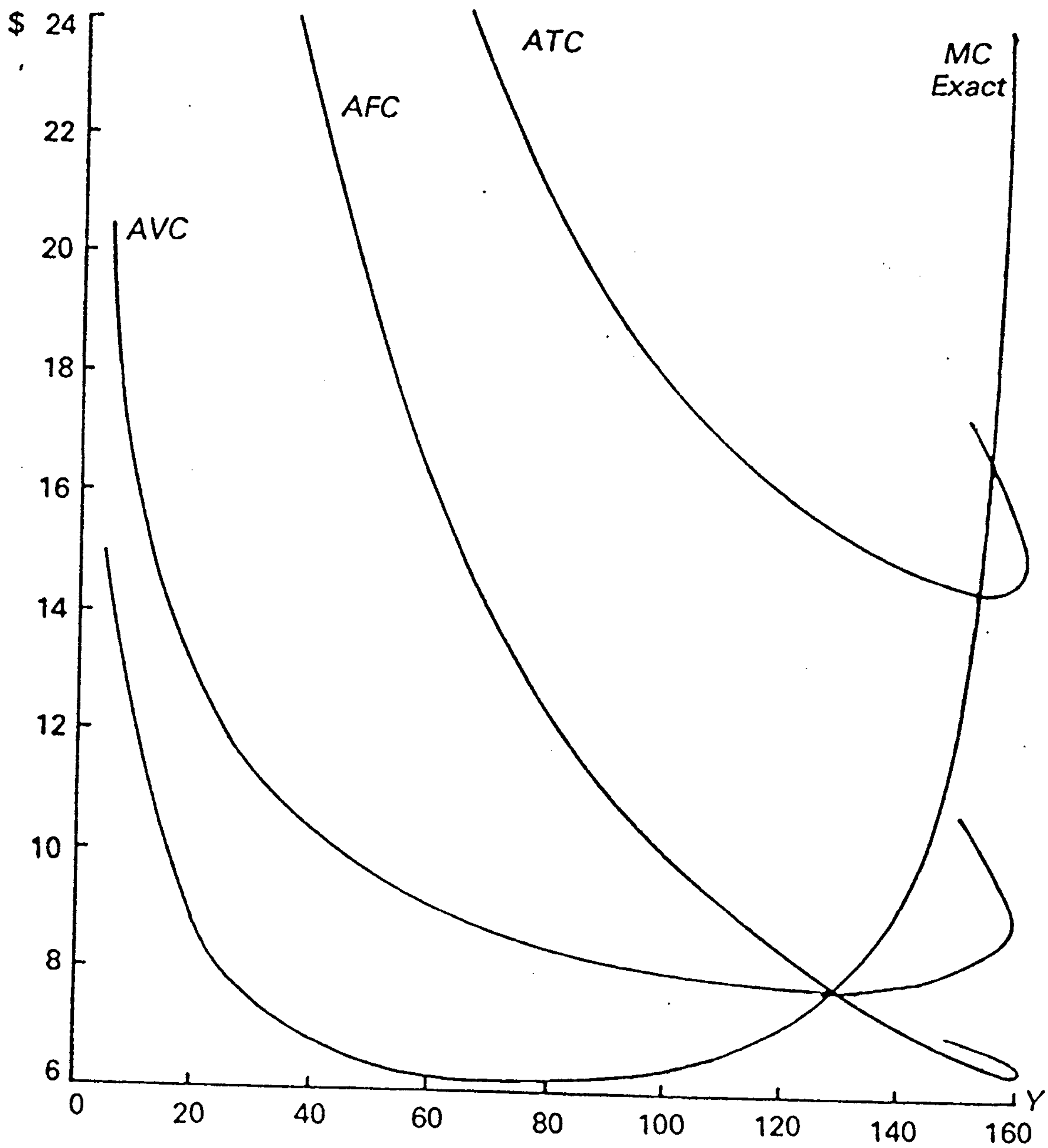


FIGURE 2–6
AVERAGE AND MARGINAL COST CURVES FOR THE CLASSICAL PRODUCTION FUNCTION

$$Y = 24.3: \quad AVC = \frac{TVC}{Y} = \frac{P_x X}{Y} = \frac{\$100 \cdot 3}{24.3} = \$12.3$$

$$Y = 87.4: \quad AVC = \frac{\$700}{87.4} = \$8.0$$

As *AFC*, *AVC* cannot be computed when output is zero.

Average variable cost is inversely related to average physical product. When *APP* is increasing, *AVC* is decreasing. When *APP* is at a maximum, *AVC* attains a minimum. When *APP* is decreasing, *AVC* is increasing. Compare the cost curves in Figure 2-6 to the production function in Figure 2-4B, from which the cost curves were derived. *APP* attains a maximum at an input level of 10, when output

is 130.0; in Figure 2-6, *AVC* is a minimum at 130.0 units of output. For output amounts between 0 and 130.0, *APP* is increasing and *AVC* is decreasing. For output levels larger than 130.0, *APP* is decreasing and *AVC* is increasing. Thus, for a production function, *APP* measures the efficiency of the variable input; for cost curves, *AVC* provides the same measure. When *AVC* is decreasing, the efficiency of the variable input is increasing; efficiency is at a maximum when *AVC* is a minimum and is decreasing when *AVC* is increasing. The relationship between *APP* and *AVC* can be demonstrated algebraically as follows:

$$AVC = \frac{TVC}{Y} = \frac{P_x X}{Y} = P_x \cdot \frac{X}{Y} = \frac{P_x}{APP}$$

because $(X/Y) = (1/APP)$.

Average total costs, *ATC*, can be computed two ways. Total costs can be divided by output or *AFC* and *AVC* can be added. *ATC* is presented in column 8 of Table 2-3 and in Figure 2-6. The shape of the *ATC* curve depends upon the shape of the production function. In Figure 2-6, *ATC* decreases as output increases from zero, attains a minimum, and increases thereafter. *ATC* is often referred to as the *unit cost* of production—the cost of producing one unit of output. The intial decrease in *ATC* is caused by the spreading of fixed costs among an increasing number of units of output and the increasing efficiency with which the variable input is used (as indicated by the decreasing *AVC* curve). As output increases further, *AVC* attains a minimum and begins to increase; when these increases in *AVC* can no longer be offset by decreases in *AFC*, *ATC* begins to rise. For the output amounts of 87.4 and 157.3, *ATC* are computed as follows:

$$Y = 87.4: \quad ATC = \frac{TC}{Y} = \frac{\$1{,}700}{87.4} = \$19.4$$

or

$$ATC = AFC + AVC = \$11.4 + 8.0 = \$19.4$$

$$Y = 157.3: \quad ATC = \frac{\$2{,}300}{157.3} = \$14.7$$

or

$$ATC = \$6.4 + \$8.3 = \$14.7$$

Marginal Cost

Marginal cost, *MC*, is defined as the change in total cost per unit increase in output. It is the cost of producing an additional unit of output. *MC* is computed by dividing the change in total costs, ΔTC, by the corresponding change in output, ΔY. Examples of marginal

cost are presented in Table 2-3 and Figure 2-6. Between the output amounts of 4.9 and 13.2, MC is computed as follows:

$$MC = \frac{\Delta TC}{\Delta Y} = \frac{\$1{,}200 - \$1{,}100}{13.2 - 4.9} = \frac{\$100}{8.3} = \$12.0$$

Between the output amounts of 130.0 and 141.9, MC is:

$$MC = \frac{\$2{,}100 - \$2{,}000}{141.9 - 130.0} = \frac{\$100}{11.9} = \$8.4$$

By definition, the only change possible in total costs is the change in variable cost, because fixed cost does not vary as output varies. Thus, $\Delta TC = \Delta TVC$. Therefore, MC could also be computed by dividing the change in total variable cost by the change in output.

Geometrically, MC is the slope of the TC curve and the TVC curve. The marginal cost of \$12 is the average of all the slopes on TC between the points $Y = 4.9$, $TC = \$1{,}100$ and $Y = 13.2$, $TC = \$1{,}200$. Within these output limits each unit of output added to total output will cost \$12 to produce. The "average" MC between the outputs of 130.0 and 141.9 is equal to \$8.4. The MC calculated in Table 2-3, column 9, should be viewed as the slope between the corresponding levels of output and not the exact slope at the indicated levels of output. The data in Table 2-3 column 10, on the other hand, show the exact MC for each corresponding level of output. The average MC and the exact MC are corollaries to the average MPP and exact MPP—Table 2-2.

The shape of the MC curve is in an inverse relationship to that of MPP. Compare the MC curve in Figure 2-6 to the MPP curve in Figure 2-4B. MPP is a maximum at 6.67 units of input; output at this point is 79.3. In Figure 2-6, MC is a minimum at 79.3 units of output. For lower levels of output, MC is decreasing while MPP is increasing. For output levels above 79.3, MPP is decreasing while MC is increasing. Algebraically, the relationship between MPP and MC can be shown as

$$MC = \frac{\Delta TC}{\Delta Y} = \frac{\Delta TVC}{\Delta Y} = \frac{P_x(\Delta X)}{\Delta Y} = P_x\frac{(\Delta X)}{(\Delta Y)} = \frac{P_x}{MPP}$$

where the change in variable costs between two output amounts, TVC, is equal to the change in the variable input used, ΔX, multiplied by the price of the input. The term $(\Delta X / \Delta Y)$ is, of course, the inverse of the MPP. Thus, the exact marginal cost figures in Table 2-3 were computed by dividing the input price, P_x, by the exact marginal product at each output level.

MC and AVC are equal when 10 units of input are used and output is 130.0; this is the same point where MPP is equal to APP. For output amounts lower than 130.0, MC is less than AVC; for higher outputs, MC is greater than AVC. As long as there is some fixed cost, MC crosses ATC at an output greater than the output at which

AVC is at the minimum and MC is equal to ATC at the latter's minimum point.

Comments On Costs

Costs are usually computed as a function of output. That is, a manager is usually interested in the total cost of producing an output or in the unit cost at a level of output. Thus, the cost curves in Figures 2-5 and 2-6 are graphed with dollars on the vertical axis and units of output on the horizontal axis. Costs are graphed as a function of input (with units of input on the horizontal axis) only in special situations.

Costs need be computed and graphed for input and output amounts only in Stages I and II of the production function; Stage III is an area in which no rational manager would produce. Stage II begins at the point where $MC = AVC$ and continues to the point where output is a maximum. In Figure 2-4, these limits are 130.0 and 159.6 units of output, inclusively. On the boundary between Stage II and III, *MPP*, as it was shown earlier, is zero. Therefore, on the same boundary, $MC = P_x/MPP = P_x/0$, which is an undefined quantity. Intuitively, the MC curve at this point would be vertical and MC would cease to have meaning. For this reason the MC data in Table 2-3 is omitted when TPP is decreasing, or when X exceeds 14 units.

Marginal cost is a widely used concept in agricultural economics. Strictly defined, it is the increase in total cost resulting from a one unit increase in output. Any other definition of marginal cost is not valid. That is, one could define a cost concept that measures the change in total cost caused by a one unit increase in input. Such a cost concept is useful but is not marginal cost; it is often called the marginal factor cost or the marginal expense of the variable input. In pure competition, the marginal factor cost is P_x.

MORE ON COST AND PRODUCTION FUNCTIONS

In practice, there are two ways researchers estimate cost functions for a farm enterprise. One way is to estimate the cost functions directly. For example, by observing costs and output data for a large sample of similar farms, the "typical" relationship between costs and output for (say) the corn enterprise or the dairy enterprise could be developed. Once the total cost functions are obtained, the other cost functions could be determined.

The second way is to estimate the cost functions directly from the production function. The production function, when known, can be used along with fixed costs and input prices to derive all cost functions.

Cost Function Known

When the total cost function can be estimated directly, then other

we will present an algebraic illustration. A total cost function with the shape or form that corresponds to a general classical production function may also be expressed as a cubic equation. For example, such a function for total cost could be

$$TC = 100 + 6Y - 0.4Y^2 + 0.02Y^3 \tag{2.6}$$

When graphed this equation would have the same general shape as the *TC* function in Figure 2-5 (it is not the equation of the *TC* curve in the graph, however). In this cost function,

$$TFC = 100 \quad \text{and} \quad TVC = 6Y - 0.4Y^2 + 0.02Y^3$$

The "100" is constant and does not change as output, *Y*, changes. The remaining terms in the function, $6Y - 0.4Y^2 + 0.02Y^3$, change with every change in output, *Y*.

From the total cost function, other costs can be derived. Thus, average variable cost would be

$$AVC = \frac{TVC}{Y} = \frac{6Y - 0.4Y^2 + 0.02Y^3}{Y}$$

$$= 6 - 0.4Y + 0.02Y^2 \tag{2.7}$$

average fixed cost would be

$$AFC = \frac{100}{Y} \tag{2.8}$$

marginal cost would be

$$MC = \frac{dTVC}{dY} = 6 - 0.8Y + 0.06Y^2 \tag{2.9}$$

Because the total cost function corresponds to the classical production function it has average variable cost and marginal cost curves that decrease, reach a minimum, and then increase. The level of output, *Y*, at which the average variable cost and marginal cost reach their minima can be calculated.

At the point where *AVC* is at the minimum its slope equals zero. Thus from (2.7)

$$\frac{dAVC}{dY} = -0.4 + 0.04Y \quad \text{or} \quad Y = 10 \tag{2.10}$$

The average variable cost, in the above example, reaches its minimum when output, *Y*, is 10 units. At this level of output the average variable cost equals marginal cost. By substituting value of 10 for *Y* in the average variable cost equation, (2.7) and marginal cost equation, (2.9) the corresponding costs, $4, are equal.

Similarily, the value of *Y* at which marginal cost is a minimum can be calculated by equating the *MC* slope to zero. From (2.9)

$$\frac{dMC}{dY} = -0.8 + 0.12Y \quad \text{or} \quad Y = 6.67 \tag{2.11}$$

Thus, the minimum of MC occurs where output equals 6.67 units and this is, as would be expected, at a lower output than the point at which AVC is a minimum.

The average total cost, ATC, reaches its minimum at an output larger than the minimum of the average variable cost because of the effect of average fixed cost on average total cost. From the above example,

$$ATC = \frac{TC}{Y} = 100Y^{-1} + 6 - 0.4Y + 0.02Y^2 \tag{2.12}$$

and

$$\frac{dATC}{dY} = -100Y^{-2} + 0.04Y - 0.4$$

or

$$\frac{-100}{Y^2} + 0.04Y - 0.4 = 0$$

Solving this equation by approximation, the value of output, Y, when the slope of the ATC curve equals zero, is 17.85 units. At this level of output ATC is at the minimum and equals the value of MC. The value for both MC and ATC is 10.83.

Finally, the slope of the MC curve at any point can be determined by substituting into (2.11). When $Y = 10$, AVC is a minimum. Substituting $Y = 10$ in (2.11) shows that MC has a positive slope at that point. The same is true where ATC is a minimum. Thus, the MC curve is increasing through those points. The properties of cost curves displayed in Figure 2-6 can be derived using calculus.

Deriving Cost Functions From Production Functions

Cost functions incorporate both the production function and the fixed and variable costs of production. Production functions are purely technical and depict only what happens to output as variable inputs are applied to a technical unit. But economic analysis of the production process requires the consideration of costs and returns. Among the many production alternatives that exist on the farm, the farmer must consider those that will be most profitable. Market forces far from the farm will help determine that profitability. In our development of the economic principles applicable to the analysis of farm enterprises, cost functions are the first step in incorporating the impact of the marketplace upon the farm enterprise. We will now develop the unique manner in which production functions and inputs prices can be combined to derive cost curves

Cost curves usually are expressed so that output, Y, is the so-called independent variable. The costs themselves represent the costs of inputs, either fixed or variable. Combining these two statements, cost curves express the cost of the fixed and variable inputs as functions of the amount of output.

Production functions, on the other hand, express output as a function of input. But input costs are input quantities multiplied by input prices. Therefore, cost functions and production functions are by nature inversely related to each other. Knowledge of one implies knowledge of the other—when input prices are known.

In symbolic notation, total variable cost has the equation

$$TVC = P_x X$$

while the production function is expressed as

$$Y = f(X)$$

But, the standard concept of the cost relationship demands that variable cost be expressed as a function of output, not input.

This dilemma is solved by expressing input, X, as a function of output, Y, for input and output amounts restricted to Stages I and II. When this is done, the resulting function is called the inverse production function and is expressed as

$$X = f^{-1}(Y)$$

where f^{-1} denotes the inverse function of the original production function f. This inverse function is not a cause-effect relationship; it does not imply output causes input. It tells the minimum amount of input needed to produce a given level of output.

When the equation for a production function is available, it may be possible to solve for the inverse production function. In any event, the shape of the inverse function can be determined on a graph. Some algebraic examples will be presented first.

A simple but very useful algebraic form for a production function, based on the parabola, is called the quadratic production function. An example is

$$Y = 8X - {}^{1}\!/_{2}X^2 \tag{2.13}$$

where Y and X are as previously defined. For this production function, APP and MPP are linear functions found as follows

$$APP = \frac{Y}{X} = 8 - {}^{1}\!/_{2}X$$

and

$$MPP = \frac{dY}{dX} = 8 - X$$

Total product takes a maximum value of 32 when $X = 8$, the same value of X for which marginal product is zero. The zone of economic

relevance for this function is $0 \leqq X \leqq 8$ with Y values that have the limits $0 \leqq Y \leqq 32$. This function along with the APP and MPP curves are shown in Figure 2-7. (Note: Figure 2-7 is not drawn to scale.)

The inverse production function for (2.13) is found using the quadratic formula. The production can be written in standard form as

$$0 = -\tfrac{1}{2}X^2 + 8X - Y$$

which, when substituted into the quadratic formula, yields

$$X = 8 - \sqrt{64 - 2Y} \quad \text{for} \quad 0 \leqq Y \leqq 32$$

Variable cost for this production function can be expressed as

$$TVC = P_x X = P_x (8 - \sqrt{64 - 2Y}), \; 0 \leqq Y \leqq 32$$

and marginal cost would be

$$MC = \frac{dTVC}{dY} = \frac{P_x}{\sqrt{64 - 2Y}}, \; 0 \leqq Y < 32$$

which is undefined when Y equals 32. Notice that MC becomes very large and will increase without limit as Y approaches 32; thus, the line $Y = 32$ becomes a vertical asymptote for MC. These curves are also presented in Figure 2-7.

An example of an algebraic production function that is easier to solve is the famous Cobb-Douglas equation.

$$Y = 6X^{1/2} \tag{2.14}$$

where the exponent of X is the elasticity of production and usually restricted to an interval between zero and one. The marginal and average product equations for this function would be

$$APP = \frac{6}{\sqrt{X}}$$

$$MPP = \frac{dY}{dX} = \frac{3}{\sqrt{X}}$$

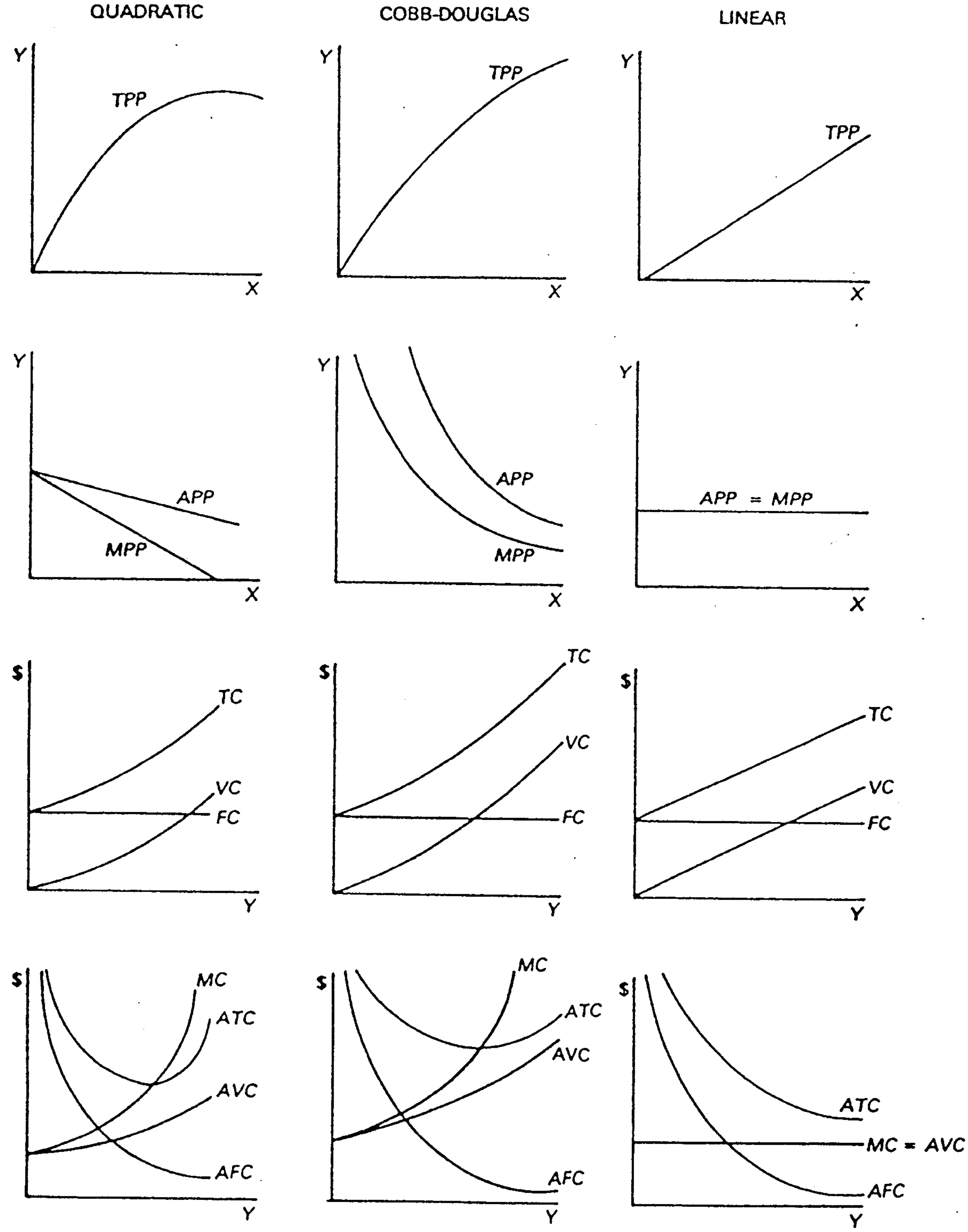
The inverse production function in this case is

$$X = \frac{1}{36} Y^2$$

so that

$$TVC = P_x X = \frac{P_x}{36} Y^2$$

$$AVC = \frac{P_x}{36} Y$$



Note: Not drawn to scale.

FIGURE 2–7
Cost Curves Derived from Three Types of Production Functions

and

$$MC = \frac{dTVC}{dY} = \frac{P_x}{18} Y$$

This function is also illustrated in Figure 2-7.

function. An example would be

$$Y = 2X \tag{2.14}$$

For this function,

$$APP = MPP = 2$$

$$X = {}^{1}/_{2}\,Y$$

$$TVC = {}^{1}/_{2}\,P_x Y$$

and

$$AVC = MC = {}^{1}/_{2}\,P_x$$

This function is also depicted in Figure 2-7.

Much of this chapter has been devoted to analysis of the classic production function and associated cost curves. As yet, the inverse function for this function has not been derived. The reason for this is that we chose a cubic equation to represent the classic function in our example, and a simple algebraic expression for the inverse of the cubic equation cannot be found. This situation sometimes happens in mathematics. The inverse function *exists* in Stages I and II of the classic function expressed as a cubic; it just cannot be expressed as a simple algebraic formula. In such cases the shape of the inverse production function and hence the variable and total cost curves can be determined visually. The shape is often all that is needed for expository purposes.

To determine the shape of the variable cost curve (or total cost curve) for any conceivable production function, hold a graph of the production function up to a mirror with output on the horizontal axis. If a mirror isn't handy, this same "upsidedown and backwards" view can be obtained from a graph. Imagine that a graph of the production function is transparent and centered on an axis that passes through the origin at a 45 degree angle. Visualizing the 45 degree axis as an axle, imagine rotating the graph 180°. The X axis and the Y axis exchange places. Further, each point on the production function that was represented by (X_o, Y_o) becomes the point (Y_o, X_o). Points falling on the 45 degree line stay in place, because it is the axis of rotation. An example of this for the classic function is depicted in Figure 2-8.

In Figure 2-8, the following points shift places as the X and Y axes are reversed.

$$(5, 3) \rightarrow (3, 5)$$

$$(10, 8) \rightarrow (8, 10)$$

$$(14, 14) \rightarrow (14, 14)$$

$$(16, 20) \rightarrow (20, 16)$$

2–3. Average total cost $ATC = 100/Y - 3Y + 4Y^2$. Calculate: (*a*) Total fixed cost. (*b*) Average variable cost when $Y = 2$. (*c*) Total variable cost when $Y = 2$. (*d*) Marginal cost when $Y = 2$. (*e*) Level of Y at which AVC is at the minimum.

2–4. The total cost function is $TC = 2Y - 2Y^2 + Y^3$. (*a*) Find the average variable cost function. (*b*) Find the level of Y where the AVC is at the minimum. (*c*) Find the marginal cost function. (*d*) Show that, at the minimum of average cost, average cost is equal to marginal cost. Plot the average cost and marginal cost curves.

2–5. The production function fitted to agronomic experimental data showing the corn response, Y, in bushels, to nitrogen, X in pounds, on a per acre basis is of the form $Y = 37 + 0.8X - 0.001X^2$. (*a*) Estimate the per acre yield of corn when $X = 0, 40, 80, 120, \ldots, 160, \ldots, 480$ pounds. (*b*) What is the *APP*, average *MPP* and exact *MPP* at the above nitrogen application rates? (*c*) At what level of nitrogen is the total yield, *TPP*, maximum? (*d*) Calculate the arc (average) and exact elasticities of production at 40 pounds of nitrogen intervals. (*e*) Plot the *TPP*, *APP*, and *MPP* curves.

2–6. Calculate the appropriate figures for the blank columns.

X	TFC	TPP	APP	MPP	TVC	AVC	MC	AFC	ATC
0	\$40.00	0	—	—	\$ 0	—	—	—	—
1	40.00	4	—	—	5.00	—	—	—	—
2	40.00	10	—	—	10.00	—	—	—	—
3	40.00	15	—	—	15.00	—	—	—	—
4	40.00	18	—	—	20.00	—	—	—	—
5	40.00	20	—	—	25.00	—	—	—	—

2–7. Consider a U-shaped AVC curve. Draw the kind of *MPP*, *APP*, and *TPP* curves you expect to find. Explain your procedure carefully.

2–8. $Y = 10X - X^2$ Find the exact or point elasticities of production when $X = 2$, 4 and 5.

2–9. Only one resource, X, is used in producing an output, Y. As X is increased, total physical product, Y, increases at a decreasing rate, reaches a maximum level and then decreases. Show graphically the relation between the total physical product curve, the marginal physical product curve, the average physical product curve and their respective cost curves.

2–10. $AVC = Y^2 - 2Y + 2$ (*a*) Derive TVC and MC equations. (*b*) At how many units of Y is AVC at the minimum? (*c*) At how many units of Y is MC at the minimum? (*d*) At how many units of Y does MC curve become the supply function for the product Y?

2–11. Draw the classical production function, *APP* and *MPP*, and carefully delineate the three stages of production.

2–12. There is some fixed cost. Does the minimum average total unit cost occur at an output less than, equal to, or greater than the output at which

2–13. What can be said about the elasticity of production, assuming the classical production function, at the output at which:

(*a*) *MPP* is at maximum
(*b*) *APP* is at maximum
(*c*) *MPP* is zero
(*d*) $MPP = APP$
(*e*) *MPP* is negative
(*f*) *MPP* is less than *APP*

2–14. For the production function $Y = f(X)$, show in general using calculus that $APP = MPP$ where *APP* is a maximum. Hint: Differentiate $f(X)/X$.

2–15. The *AFC* curve is called a rectangular hyperbola, which means that any rectangle with two sides on the axes, a corner on the curve, and sides parallel to the axes will have a constant area. What is the implication of that for our study of cost curves? Hint: Would a similar interpretation hold for any rectangle drawn under the *AVC* curve? What do the areas of these rectangles represent?

2–16. Use calculus to show that the *AFC* curve always has a negative slope. Will the slope of *ATC* curve always equal the sum of the slopes of the *AFC* and *AVC* curves?

2–17. Show that if all other costs remain the same, the output level at which *ATC* is a minimum will depend on the magnitude of *TFC*. Will an increase in *TFC* increase or decrease the output level at which *ATC* is a minimum?

2–18. The graphs in Figure 2-7 are not drawn to scale. Choose numerical amounts for P_x and *FC*, derive all cost curves and plot the graphs to scale using the production functions given in the text.

2–19. Plot the production functions used in Figure 2-7 accurately on graph paper. On the same graph, plot the inverse production function in each case using the technique described in the text. Be sure to use the same scale on both axes.

ENDNOTES

1. Earl O. Heady, John T. Pesek, William G. Brown, and John P. Doll, "Crop Response Surfaces and Economic Optima in Fertilizer Use," Chapter 14 in *Agricultural Production Functions*, Iowa State University Press: Ames, 1961.
2. John M. Cassels, "On the Law of Variable Proportions," *Readings in the Theory of Income Distribution*, Philadelphia: The Blakiston Company, 1951, Chapter 5.
3. The farmer is assumed to be operating in pure competition. The input price is, therefore, constant regardless of the amount purchased. A brief discussion of pure competition is included in Chapter 3.

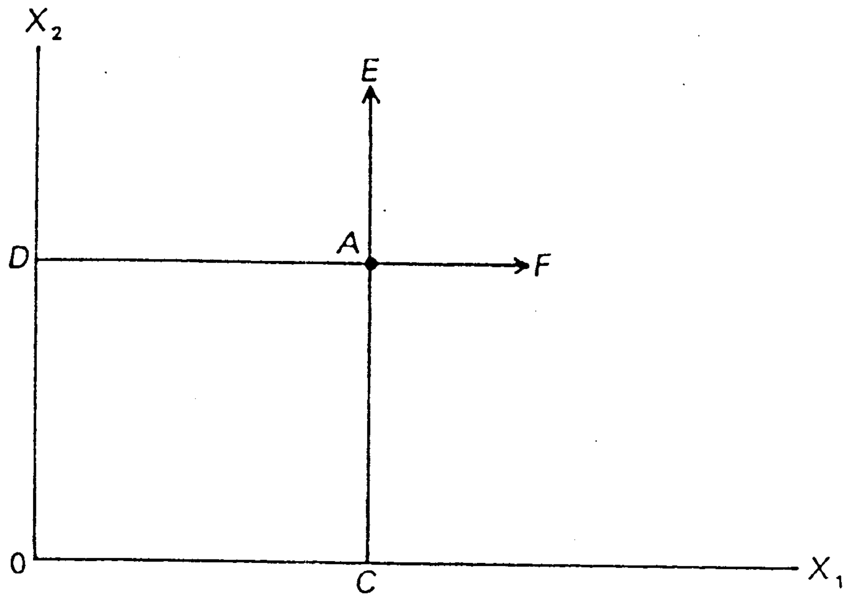


FIGURE I–11

EXPONENTS

Use of the Cobb-Douglas function requires some manipulation of exponents. In this section we apply the rules of differentiation to exponential functions and review some rules of exponents.

If

$$Y = -3X^{1/3},$$

$$\frac{dY}{dX} = -3 \cdot \frac{1}{3} X^{1/3-1} = -1 \cdot X^{-2/3} = -X^{-2/3}$$

It may be useful now to recall the meanings of negative exponents and fractional exponents. (In the expression HX^K, K is the exponent of X—the power to which X is raised.)

$X^{1/a}$ means the ath root of X. Thus, $X^{1/2}$ is the square root of X, and $X^{1/3}$ the cube root. $X^{a/b}$ may be considered the bth root of the quantity X raised to the ath power, or the ath power of the bth root of X. Thus,

$$X^{a/b} = [X^a]^{1/b} = [X^{1/b}]^a \tag{9}$$

We can see that the three expressions in (9) are equivalent if we recall that

$$[X^c]^d = X^{cd}$$

If this sounds only vaguely familiar, test it. For instance,

$$[2^3]^2 = 8^2 = 64 = 2^{3 \cdot 2} = 2^6$$

Of course, it is also true that

$$[X^c]^d = [X^d]^c = X^{cd} \tag{10}$$

Notice that (9) is like (10), except that a fraction is involved. A negative exponent refers to the operation of moving a factor across the dividing line of a fraction thus

$$X^{-a} = \frac{1}{X^{a}}$$

and

$$\frac{1}{X^{-a}} = X^{a}$$

Seeing the nature of a negative exponent enables us to expand the range of problems that we can handle with the rule for differentiating exponents. For instance, let

$$Y = \frac{4}{X^3} \tag{11}$$

At first glance it may appear that the rule cannot be used. We must have the factor in X in the numerator. But (11) can be written

$$Y = 4X^{-3}$$

so that

$$\frac{dY}{dX} = -12X^{-4}$$

FINDING EXTREME POINTS

Functions Of One Variable

Derivatives are often used to find the points at which functions take maximum or minimum values, called extreme points. When a function of one independent variable takes a maximum or a minimum at a point, its tangent at that point will have a slope of zero. Thus, when searching for extremes, we can equate the derivative of a function to zero and solve, finding one or more particular values of X. The function will have tangents with zero slopes at these points; thus, they should lead to the extremes, if any exist. Points where the derivative is zero are called critical points. For example, consider the function $Y = X^2$ which has the derivative $2X$. But $2X = 0$ will hold only for $X = 0$, so $X = 0$ and $Y = 0$, the point (0, 0), will be an extreme point. Now, $Y = X^2$ will increase without limit as X either increases or decreases, so the critical point will be a minimum. Thus, in each case, the value of X for which a derivative is zero must be examined with care. While it is true that the derivative will always be zero at maximum (or minimum), it is not always true that a maximum (or minimum) will occur at an X value when the

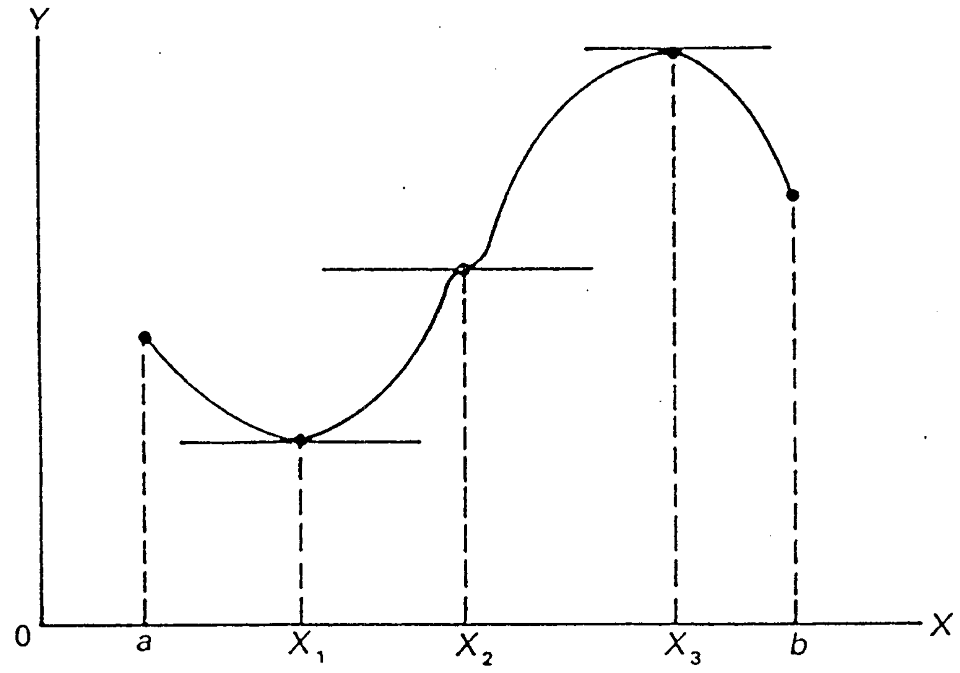


FIGURE I–12

derivative is zero. All extreme points are critical points but all critical points are not extreme points.

The possible cases are shown in Figure I-12. The curve in the graph has three tangents with zero slope in the interval (a, b). At X_1, a minimum occurs: X_3 is a maximum. But at X_2 the function takes neither a maximum nor a minimum; X_2 is a critical point at which the tangent crosses the function.

Graphing the curve around a critical point will usually determine which of the three cases has occurred. We could also examine the slope of the curve on each side of the critical points using the derivative equation. Study of Figure I-12 would suggest the following table:

POINT	EXTREME?	SLOPE ON LEFT	SLOPE ON RIGHT
X_1	minimum	−	+
X_3	maximum	+	−
X_2	none	+	+

But the signs derived in the table suggest that if a minimum is to occur, the slope (derivative) of the function must be increasing through the critical point. The slope of a function is negative at the left of a minimum value, zero at the minimum value, and positive on the right. For a maximum, the reverse is true. If the derivative increases through a critical point, a minimum exists; if it decreases through a critical point, a maximum exists. Recall that the *MPP* is decreasing and equal to zero when *TPP* is a maximum.

All of this suggests that we examine the slope of the derivative at the critical points. Not just the *value* of the derivative but also

its *slope.* To do this we take the derivative of the derivative; such a derivative is called the *second* derivative. What we have been calling THE derivative we will now call the *first* derivative. The second derivative is obtained from the equation of the first derivative using all the same concepts described above.

To summarize our rules for finding and evaluating critical points: If the value of the first derivative is zero and the value of the second derivative is negative at a critical point, the critical point is a maximum. If the value of the first derivative is zero and the value of the second derivative is positive, the critical point is a minimum. If the value of the second derivative is zero (as at point X_2 in Figure I-12), further examination will be required.

For example, consider the function $Y = X^2$, then

$$\frac{dY}{dX} = 2X \quad \text{and} \quad \frac{d}{dX}\left(\frac{dY}{dX}\right) = \frac{d^2 Y}{dX^2} = 2$$

where $d^2 Y/dX^2$ is used to denote the second derivative of $Y = X^2$ with respect to X.

But

$$\frac{dY}{dX} = 2X = 0 \quad \text{for} \quad X = 0 \quad \text{and} \quad \frac{d^2 Y}{dX^2} = 2 > 0 \text{ for all } X$$

implies that at the critical point, $X = 0$, the function has a minimum.

As a second example, consider the function

$$Y = 8X - X^2$$

then

$$\frac{dY}{dX} = 8 - 2X \quad \text{and} \quad \frac{d^2 Y}{dX^2} = -2$$

The first derivative is zero at $X = 4$ and the second derivative is always negative. The critical point that occurs at $X = 4$ is a maximum.

As a final example, consider the production function equation used in Chapter 2:

$$Y = 3X + 2X^2 - 0.1X^3$$

$$\frac{dY}{dX} = 3 + 4X - 0.3X^2$$

$$\frac{d^2 Y}{dX^2} = 4 - 0.6X$$

The first derivative takes the value of zero at two points, $X = -0.7$ and $X = 14$. Evaluating the second derivative at these points

$$X = -0.7{:}\ 4 + 0.6(.7) = 4 + 0.4 = 4.4 > 0, \text{ minimum;}$$

$$X = 14{:}\ 4 - 0.6(14) = 4 - 8.4 = -4.4 < 0, \text{ maximum}$$

Note that for ease of exposition in Chapter 2, *MPP* and *APP* for this function were drawn as if the minimum occurred at X equals zero. When the first and second derivatives are both zero, the tests must be carried further. Higher derivatives should be evaluated at the critical point. If the first nonzero derivative is even,

$$\frac{d^4Y}{dX^4}, \quad \frac{d^6Y}{dX^6}, \quad \frac{d^8Y}{dX^8}, \quad \text{etc.,}$$

then the sign of that derivative can be used to test for a maximum or minimum, exactly as above. If the first nonzero derivative is odd, the tangent crosses the function at that critical point. For practice, consider $Y = X^5$, $Y = X^6$, $Y = 3X^2 - X^6$, etc.

Functions Of Two Variables

The critical points for functions of two variables may also be located using derivatives. When a surface takes a maximum or minimum value, then both first partial derivatives will have values of zero at that point. Regarding the derivatives as slopes along the sub-production functions that parallel the two axes, the maximum on the surface will occur only at a point for which both sub-production functions attain their maximums simultaneously. Rather than a tangent line, the surface will have a tangent plane that touches the surface at the critical point and has slopes in each direction determined by the values of the derivatives. At each critical point the tangent plane will be flat—horizontal in both directions.

Therefore, a surface such as $Y = f(X_1, X_2)$ will have critical points for values of X_1 and X_2 where

$$\frac{\partial Y}{\partial X_1} = 0 \quad \text{and} \quad \frac{\partial Y}{\partial X_2} = 0$$

Examples are presented in Appendix II under the heading "Unconstrained Maximization" and will not be repeated here.

As before, the critical points may be maximums, minimums, or a third alternative, called now a saddlepoint. There is a second derivative test that is easily used but not readily justified intuitively. We will state the conditions without extensive discussion.

When the first partial derivatives are zero at a point and the second partial derivatives, when evaluated, take the following signs

$$\text{(a)}\ \frac{\partial^2 Y}{\partial X_1^2} < 0, \quad \frac{\partial^2 Y}{\partial X_2^2} < 0 \quad \text{and} \quad \left(\frac{\partial^2 Y}{\partial X_1^2}\right)\left(\frac{\partial^2 Y}{\partial X_2^2}\right) - \left(\frac{\partial^2 Y}{\partial X_1 \partial X_2}\right)^2 > 0$$

then the critical point is a maximum. When

$$\text{(b)}\ \frac{\partial^2 Y}{\partial X_1^2} > 0, \quad \frac{\partial^2 Y}{\partial X_2^2} > 0 \quad \text{and} \quad \left(\frac{\partial^2 Y}{\partial X_1^2}\right)\left(\frac{\partial^2 Y}{\partial X_2^2}\right) - \left(\frac{\partial^2 Y}{\partial X_1 \partial X_2}\right)^2 > 0$$

then the critical point is a minimum. The notation can best be explained with an example. Consider the equation

$$Y = 6X_1 - X_1^2 + 8X_2 - 1/2X_2^2 + X_1X_2$$

then

$$\frac{\partial Y}{\partial X_1} = 6 - 2X_1 + X_2 \quad \text{and} \quad \frac{\partial Y}{\partial X_2} = 8 - X_2 + X_1$$

The second partial derivatives are the derivatives of the first partials taken with respect to the same variables. Thus,

$$\frac{\partial^2 Y}{\partial X_1^2} = -2 \quad \text{and} \quad \frac{\partial^2 Y}{\partial X_2^2} = -1$$

The third type of second partial derivative is called a crosspartial and is

$$\frac{\partial}{\partial X_1}\left(\frac{\partial Y}{\partial X_2}\right) \quad \text{or} \quad \frac{\partial}{\partial X_2}\left(\frac{\partial Y}{\partial X_1}\right)$$

which is rewritten

$$\frac{\partial^2 Y}{\partial X_1 \partial X_2} \quad \text{or} \quad \frac{\partial^2 Y}{\partial X_2 \partial X_1}$$

Therefore

$$\frac{\partial^2 Y}{\partial X_1 \partial X_2} = 1 \quad \text{and} \quad \frac{\partial^2 Y}{\partial X_2 \partial X_1} = 1$$

To find the crosspartial, the derivative is taken first with respect to one independent variable and then the second derivative is taken with respect to the other independent variable. For continuous, differentiable functions, the order of differentiation is not important because the result will be the same.

We now apply the tests to our example. The first partial derivatives are set equal to zero and solved simultaneously. The equations to be solved are

$$-2X_1 + X_2 = -6$$
$$X_1 - X_2 = -8$$

and only one solution, $X_1 = 14$ and $X_2 = 22$, exists. A critical point will exist at these input values. The second derivative test gives

$$\frac{\partial^2 Y}{\partial X_1^2} = -2 < 0 \quad \text{and} \quad \frac{\partial^2 Y}{\partial X_2^2} = -1 < 0$$

and

$$\left(\frac{\partial^2 Y}{\partial X_1^2}\right)\left(\frac{\partial^2 Y}{\partial X_2^2}\right) - \left(\frac{\partial^2 Y}{\partial X_1 \partial X_2}\right)^2 = (-2)(-1) - (1)^2 = 1 > 0$$

The critical point is a maximum.

When a critical point of a function of two variables is neither a maximum nor a minimum, it is called a saddlepoint. If one second derivative is positive and the other negative, then the surface attains a minimum on a line parallel to one axis and a maximum on a line parallel to the other axis. The shape of this surface will appear much like a saddle, with the critical point located in the saddle seat. The minimum will occur on a line paralleling the length of the "horse"; the maximum will occur on a line perpendicular to the first line and extending to the right and left of the "horse." This type of saddlepoint is easily detected because the second partial derivatives are of opposite sign. A more subtle saddlepoint may also occur. Using a production function as an example, suppose both inputs display diminishing returns when one is increased while the other is held constant, but that there are increasing returns out an expansion path. Such a function will have a critical point that is a saddlepoint. Using the quadratic equation presented above, we can change the response by increasing the coefficient on the interaction term, X_1X_2, to (say) 2. Then we have

$$Y = 6X_1 - X_1^2 + 8X_2 - 1/2X_2^2 + 2X_1X_2$$

This function displays diminishing returns to each input and will have convex isoquants. But the test using the second partial derivatives is

$$\left(\frac{\partial^2 Y}{\partial X_1^2}\right)\left(\frac{\partial^2 Y}{\partial X_1^2}\right) - \left(\frac{\partial^2 Y}{\partial X_1 \partial X_2}\right)^2 = (-2)(-1) - (2)^2 = -2 < 0$$

As input use increases the positive interaction between the variables counteracts the diminishing returns to each input. For example, on an expansion path where $X_1 = X_2$,

$$Y = 6X_1 - X_1^2 + 8X_1 - 1/2X_1^2 + 2X_1X_1 = 14X_1 + 1/2X_1^2$$

and, as might be expected, the function has no critical points in the positive orthant. The coefficient on the interaction term can be adjusted to yield constant returns along an expansion path. Let the coefficient on X_1X_2 be equal to $\sqrt{2}$. Then the partial derivative test presented in the preceding paragraph will be zero and constant returns will exist along the expansion path $X_2 = \sqrt{2X_1}$.

SUGGESTED READINGS

Allen, R. G. D. *Mathematical Analysis for Economists*. St. Martin's Press, 1956. Chapters 5, 6, 7 and 8.

Carlson, R. J. *Mathematics for Microeconomic Theory*. Grid, Inc., 1976, Chapters 4, 9, and 10.

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Appendix II

INTRODUCTION TO NONLINEAR PROGRAMMING

Linear programming is a special case of a more general set of programming techniques. The unique aspect of linear programming is, of course, obvious. Both the objective and the constraint function are linear. In general, this does not need to be true; the world is not necessarily linear.

This section presents an introduction to nonlinear programming. A very general introduction to nonlinear techniques, such as that contained in Hadley, requires advanced mathematics. However, an introduction can be presented using only the techniques of elementary calculus, provided the reader will accept intuitive explanations of some rather difficult mathematical concepts.

The outline for the remainder of this appendix will be as follows: First, the concepts of unconstrained maximization will be reviewed. Second, maximization subject to a constraint will be demonstrated again, but it will be used to develop the Lagrangean multiplier technique. The dual will be presented in a more general form and applied to illustrate the dual solution in linear programming. Third, inequality constraints will be introduced using a series of examples. These lead to what is known as the *Kuhn-Tucker* theorems, although those theorems will not be presented in their general form. Finally, an example of a nonlinear programming problem will be illustrated using calculus.

Unconstrained Maximization

The production function presented in Chapter 4 was the quadratic (parabolic) function

$$Y = 18\,X_1 - X_1^2 + 14\,X_2 - X_2^2$$

And, although not discussed at the time, it is clear that the nonnegativity restrictions $Y \geqq 0$, $X_1 \geqq 0$ and $X_2 \geqq 0$ must be applied to this equation if it is to truly represent a production function. Thus, even though we will call

economic theory will always dictate some type of constraint or restriction to insure that mathematical equations will conform to economic reality. With that caveat, the unconstrained maximum can be found by setting the first partial derivatives of the production function, taken with respect to the two inputs, equal to zero and solving the resulting two equations simultaneously. Thus the input amounts that result in the maximum yield are found by solving

$$\frac{\partial Y}{\partial X_1} = 18 - 2X_1 = 0, \quad \text{or} \quad X_1 = 9$$

$$\frac{\partial Y}{\partial X_2} = 14 - 2X_2 = 0, \quad \text{or} \quad X_2 = 7$$

and the resulting maximum yield, found by substituting these input values back into the production function, is 130. In this case, because X_1 and X_2 do not occur in both equations, simultaneous solution of the equations is not necessary. As can be seen from the graphs in Chapter 4, this solution is a maximum.[1] Inputs and outputs are positive.

Farmers may be interested in maximizing profit rather than output. The profit equation is expressed as Profit = $TR - TC$ or, for our example

$$\text{Profit} = P_y Y - (P_{x_1} X_1 + P_{x_2} X_2) - TFC$$

where $Y = H(X_1, X_2)$ and TFC, fixed cost, is a constant. Maximizing this with respect to the inputs results in

$$\frac{\partial\,\text{Profit}}{\partial X_1} = P_y \cdot \frac{\partial H}{\partial X_1} - P_{x_1} = 0 \quad \text{or} \quad VMP_{x_1} = P_{x_1}$$

$$\frac{\partial\,\text{Profit}}{\partial X_2} = P_y \cdot \frac{\partial H}{\partial X_2} - P_{x_2} = 0 \quad \text{or} \quad VMP_{x_2} = P_{x_2}$$

Referring back to the example in Chapter 5, the price of output was assumed to be \$0.65 per unit, the price of X_1 was \$9, and the price of X_2 was \$7. Inserting these prices and the derivatives above in the maximizing equations gives

$$\frac{\partial\,\text{Profit}}{\partial X_1} = \$0.65(18 - 2X_1) = \$9, \quad \text{or} \quad X_1 = 2.1$$

$$\frac{\partial\,\text{Profit}}{\partial X_2} = \$0.65(14 - 2X_2) = \$7, \quad \text{or} \quad X_2 = 1.6$$

The resulting yield would be 53.17, quite different from the unconstrained maximum output. The resulting maximum profit is

$$(\$0.65)(53.17) - \$9(2.1) - \$7(1.6) - FC = \$4.46 - TFC$$

Maximizing Subject To A Constraint

To begin this section, we will work with a familiar example. Consider the problem in Chapter 4, using the same production function as above. When $P_{x_1} = \$2$ and $P_{x_2} = \$3$, a study of Figure 4-10 shows that the least cost combination of X_1 and X_2 needed to produce a yield of 105 is 6.2 and 2.8, respectively. We will work this as a maximizing problem, assuming that $\$20.80 = \$2 \cdot 6.2 + \$3 \cdot 2.8$ of capital is available.

The constrained maximization problem for this example is to maximize revenue subject to the budget constraint, given the production function, prices, and the total available capital. The total revenue function, $P_y Y$, is the objective function and the total outlay function, $\$2 X_1 + \$3 X_2 = \$20.80$ is the constraint. Geometrically, the problem is to find the point on the isocost line that produces the maximum revenue. Whereas the unconstrained maximum can be *any* input combination, the constrained maximum *must* be one of the points on the isocost line. Thus, the potential set of solutions is constrained at the outset of the problem.

The most direct solution to this problem is to solve the constraint equation for one variable as a function of the other. For example, the budget constraint can be expressed as

$$X_1 = 10.4 - 1.5X_2$$

which is the equation of the isocost line in Figure 4-10. This form of the constraint can then be substituted directly into the total revenue function. When the unconstrained total revenue, TR, is given by

$$TR = P_y \cdot Y = \$0.65\,(18X_1 - X_1^2 + 14X_2 - X_2^2)$$

then constrainted total revenue, TR_c, will be given by

$$TR_c = \$0.65[18(10.4 - 1.5X_2) - (10.4 - 1.5X_2)^2 + 14X_2 - X_2^2]$$

$$TR_c = \$0.65[79.04 + 18.2X_2 - 3.25X_2^2]$$

The equation in brackets represents the equation of the production function restricted to the set of points that lie on the isocost line for \$20.80.[2] Note that the price of output will not affect the quantities of X_1 and X_2 that will ultimately be the solution; it only affects the magnitude of the resulting total revenue. The portion of the production function represented parametrically within the brackets is the equation of the parabola stretching from points A and B in Figure 4-11.

With this as background, the problem is to maximize revenue subject to the total outlay constraint. To do this, set the derivative of the constrainted total revenue to zero as follows

$$\frac{\partial TR_c}{\partial X_2} = \$0.65(18.20 - 6.50\,X_2) = 0$$

so

$$X_2 = 2.8$$

and

$$X_1 = 10.4 - 1.5 \cdot 2.8 = 6.2$$

The maximum revenue that can be attained when \$20.80 is available is \$0.65 · 105 = \$68.25 and profit, before fixed cost, is \$47.45.

The above technique can be represented symbolically. In general, we have the objective function, $Y = f(X_1, X_2)$, and the constraint function $g(X_1, X_2) = b$. Using arrows to represent causality and the function name, g or f, to show the functional relationship, we would have in the unconstrained case

$$\begin{tikzcd} & X_1 \arrow[dl, "f"'] \\ Y & \\ & X_2 \arrow[ul, "f"] \end{tikzcd}$$

where X_1 and X_2 both influence Y, but are independent of each other. X_1 and X_2 are free to take any values whatsoever. In the constrained case, we have

$$\begin{tikzcd} & X_1 \arrow[dl, "f"'] \\ Y & \\ & X_2 \arrow[ul, "f"] \arrow[uu, "g"'] \end{tikzcd}$$

where X_1 and X_2 are connected by the budget constraint. Increasing X_2 causes a reduction in X_1 because only \$20.80 is available. Thus, X_1 and X_2 are no longer independent variables in the production function. When the constraint function, g, can be solved for X_1 as a function of X_2, say $X_1 = h(X_2)$, then the objective function can be expressed (in constrained form) as a function of X_2 only. Thus

$$Y_c = f[h(X_2), X_2] = F(X_2)$$

and the procedure above was to maximize Y_c as a function of X_2. But, when two so-called independent variables are related, the objective function can be differentiated with respect to the causal variable. Assuming the isocost function expresses X_1 as a function of X_2, then for $Y_c = f(X_1, X_2)$ we would have

$$\frac{dY_c}{\partial X_2} = \frac{\partial f}{\partial X_1} \cdot \frac{dX_1}{dX_2} + \frac{\partial f}{\partial X_2}$$

where, in diagrammatic form, the rates of change may be expressed as

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$$\begin{tikzcd} & X_1 \arrow[dl, "\frac{\partial f}{\partial X_1}"'] & \\ Y & & X_2 \arrow[ul, "\frac{dX_1}{dX_2}"'] \arrow[ll, "\frac{\partial f}{\partial X_2}"] \end{tikzcd}$$

Thus, a change in X_2 stimulates a direct change in Y, expressed as

$$\frac{\partial f}{\partial X_2}$$

and an indirect change in Y, expressed as

$$\frac{\partial f}{\partial X_1} \cdot \frac{dX_1}{dX_2}$$

In the second case, a change in X_2 necessitates a change in X_1 which causes a change in Y. Returning to our example of this section, we substitute the derivatives directly into the above equation

$$\frac{dY_c}{dX_2} = \$0.65[(18 - 2X_1) \cdot (-1.5) + (14 - 2X_2)]$$

where

$$X_1 = 10.4 - 1.5\, X_2$$

so that

$$\frac{dY_c}{dX_2} = \$0.65(18.20 - 6.50\, X_2)$$

Setting this equation to zero and solving will again give the same answers.

Direct substitution of the constraint function into the objective function depends upon being able to solve the constraint function explicitly for one input as a function of the other; that is, expressing $g(X_1, X_2) - b = 0$ as $X_1 = h(X_2)$. Sometimes this is not possible even though the function h theoretically exists. Sometimes it is possible but the function h and its derivatives are complex, to say the least. In such cases a more general solution is available using implicit differentiation. Thus, we know

$$g(X_1, X_2) = b$$

and we also know that dX_1/dX_2 exists but we can't find it directly. But by differentiating g implicitly, regarding X_1 as a function of

X_2, we get

$$\frac{\partial g}{\partial X_1} \cdot \frac{dX_1}{dX_2} + \frac{\partial g}{\partial X_2} = 0$$

since $\partial b / \partial X_2 = 0$. Solving this we find that

$$\frac{dX_1}{dX_2} = - \frac{\dfrac{\partial g}{\partial X_2}}{\dfrac{\partial g}{\partial X_1}}$$

For our revenue maximizing example, recall that

$$g(X_1,X_2) = \$2X_1 + \$3X_2 = \$20.80$$

but

$$\frac{\partial g}{\partial X_1} = 2, \quad \frac{\partial g}{\partial X_2} = 3$$

so that

$$\frac{dX_1}{dX_2} = - \frac{3}{2} = -1.5$$

which agrees with the solution we obtained by solving g for X_1 and differentiating directly. Finally, we have the derivative of the constrained objective equation

$$\frac{dY_c}{dX_2} = \frac{\partial f}{\partial X_1} \cdot \frac{dX_1}{dX_2} + \frac{\partial f}{\partial X_2} = 0$$

which we set equal to zero because we want to maximize the constrained value. But we can now express dX_1/dX_2 as the ratio of the partial derivatives of g so we have

$$\frac{dY_c}{\partial X_2} = \frac{\partial f}{\partial X_1} \left(- \frac{\dfrac{\partial g}{\partial X_2}}{\dfrac{\partial g}{\partial X_1}} \right) + \frac{\partial f}{\partial X_2} = 0$$

The beauty of this is that all derivatives are obtained by partial differentiation of functions we know. We do not have to explicitly solve the constraint for one variable as a function of the others or attempt to differentiate it directly.

In summary, our solution to the maximization problem subject to the constraint is given by

$$\frac{\partial f}{\partial X_2} - \frac{\partial f}{\partial X_1}\left(\frac{\dfrac{\partial g}{\partial X_2}}{\dfrac{\partial g}{\partial X_1}}\right) = 0$$

and

$$g(X_1, X_2) = b$$

where the first equation gives the marginal conditions and the second determines the value of X_1, given X_2. That is, it insures the constraint will be met. But remember that we have not substituted directly to obtain the marginal conditions (the first equation). Hence, X_1 may occur in that equation. Simultaneous solution of the two equations will insure the correct result. For our maximization example, these equations give

$$0.65(18 - 2X_1) - 0.65(14 - 2X_2)\left(\frac{3}{2}\right) = 0$$

$$\$2X_1 + \$3X_2 = \$20.80$$

which has the solution $X_1 = 6.2$ and $X_2 = 2.8$

The equations we just derived are often rewritten

$$\frac{\dfrac{\partial f}{\partial X_2}}{\dfrac{\partial g}{\partial X_2}} = \frac{\dfrac{\partial f}{\partial X_1}}{\dfrac{\partial g}{\partial X_1}}$$

$$g(X_1, X_2) = b$$

The resulting values must then be substituted into the objective function to determine the value of the constrained maximum.

Minimizing Subject To A Constraint

We have just solved a constrained maximization problem several different ways. Actually, all the methods presented were the same in concept—only the method differed. The same techniques apply to minimizing subject to a constraint. We will now investigate a constrained minimization problem.

Consider the problem depicted in Figure 4-10. The economic problem is to determine the least-cost combination of inputs for an output of 105. The production function is the same one used throughout this section and the prices of the inputs are $P_{x_1} = \$2$ and $P_{x_2} = \$3$. Mathematically, the problem is to minimize

$$TVC = \$2X_1 + \$3X_2$$

subject to

$$105 = 18X_1 - X_1^2 + 14X_2 - X_2^2$$

The production function, with Y set equal to 105, is now the constraint. We cannot select *any* combination of inputs; we can only select from those combinations that will produce 105 units of output. From those, we will select the combination that costs the least.

The direct method of solution would require that the constraint be solved explicitly to express (say) X_2 as a function of X_1 and that this function be substituted directly into the variable cost function. But the explicit solution of the production function in this case would be the isoquant equation, which is a rather complex expression with an algebraically messy derivative.

However, the derivatives of the production function are easily found and, in this case, simple algebraic forms. Thus, we can determine the solution by recalling that the conditions for a minimum are

$$\frac{\dfrac{\partial f}{\partial X_2}}{\dfrac{\partial g}{\partial X_2}} = \frac{\dfrac{\partial f}{\partial X_1}}{\dfrac{\partial g}{\partial X_1}}$$

$$g(X_1, X_2) = b$$

which in this example will be

$$\frac{3}{14 - 2X_2} = \frac{2}{18 - 2X_1}$$

$$18X_1 - X_1^2 + 14X_2 - X_2^2 = 105$$

because f is now the variable cost equation and g is the production function with b equal to 105. (The first equation represents the marginal conditions presented in Chapter 4. Thus, we have again derived the marginal concepts—but in a different manner.) The first equation can be solved for X_1 to give

$$X_1 = \frac{13}{3} + 2/3X_2$$

This is, in fact, the isocline equation for $P_{x_1} = \$2$ and $P_{x_2} = \$3$. This equation can then be substituted into the constraint (production function with Y equal to 105) and the resulting expression solved using the general solution for the quadratic formula. The answers are $X_1 = 6.2$ and $X_2 = 2.8$, as in Figure 4-10.

The question the reader might ask is how we know the extreme point is a minimum rather than a maximum? This comes from the nature of the problem. In this case, the production function chosen has isoquants convex to the origin so that the extremal must be a minimum. More generally, the sufficient conditions must be examined to insure the minimum.

Notice that we have now found the solution to this problem two different ways. First, we maximized revenue subject to a budget constraint of $20.80. This led to a solution that was equivalent to maximizing yield subject to the constraint because P_y did not affect the solution. Second, we found the same answer by minimizing variable cost subject to an output constraint of 105.

The Lagrangean Function

The problems presented previously are often set up in a form known as a *Lagrangean Function.* The Lagrange technique is more general in many respects and yields additional information of value. In particular, it enables us to derive the dual solution and the shadow prices of linear programming.

We have the objective function

$$Y = f(X_1, X_2)$$

subject to the constraint

$$g(X_1, X_2) = b$$

which has the maximizing (minimizing) solution(s) at values of X_1 and X_2 where

$$\frac{\partial f}{\partial X_2} - \frac{\partial f}{\partial X_1}\left(\frac{\dfrac{\partial g}{\partial X_2}}{\dfrac{\partial g}{\partial X_1}}\right) = 0$$

and

$$g(X_1, X_2) = b$$

But notice that the first condition can be written

$$\frac{\partial f}{\partial X_2} - \left(\frac{\dfrac{\partial f}{\partial X_1}}{\dfrac{\partial g}{\partial X_1}}\right)\frac{\partial g}{\partial X_2} = 0$$

and it is true by definition that

$$\frac{\partial f}{\partial X_1} - \left(\frac{\dfrac{\partial f}{\partial X_1}}{\dfrac{\partial g}{}}\right)\frac{\partial g}{\partial X_1} = 0$$

so we proceed to define a new variable λ as

$$\lambda = \frac{\dfrac{\partial f}{\partial X_1}}{\dfrac{\partial g}{\partial X_1}}$$

so that the necessary conditions for an extreme value can be written

$$\frac{\partial f}{\partial X_1} - \lambda \frac{\partial g}{\partial X_1} = 0$$

$$\frac{\partial f}{\partial X_2} - \lambda \frac{\partial g}{\partial X_2} = 0$$

$$g(X_1, X_2) - b = 0$$

We now have three derivatives expressed in three variables (X_1, X_2, λ). Can we find a function for which these equations are the first derivatives? The answer is yes. The function, known as the Lagrangean Function, is

$$f(X_1, X_2, \lambda) = f(X_1, X_2) - \lambda[g(X_1, X_2) - b]$$

To maximize (or minimize) the function f subject to g, set up the Lagrangean Function, equate its partial derivatives with respect to X_1, X_2, and λ to zero, and solve them simultaneously. The partial derivatives are, of course, the three equations presented above in this paragraph and do therefore determine the solution. The Lagrangean multiplier is known as λ.

The interpretation of λ. The Lagrangean multiplier, λ, that seemed to be conjured out of thin air, in fact has a legitimate and very useful interpretation in economics. λ measures the amount the objective function would increase when the constraint value, b, is increased. In terms of derivatives,

$$\frac{df(X_1,X_2)}{db} = \lambda$$

This interpretation is arrived at intuitively as follows: The values of X_1, X_2, and Y at the constrained optimum are determined by the amount of the constraint. When maximizing revenue subject to the variable cost constraint, b was \$20.80. When minimizing variable cost subject to the yield constraint, b was 105. In the first instance, if we increased b to \$21.80, how much would total revenue increase? In the second, how much would variable cost increase if output were increased to 106? λ provides a numerical answer to these questions.

Conceptually, the solution is obtained by regarding b as a variable rather than a constant. Increasing b will permit increases in X_1 and

X_2 which will in turn cause increases in total revenue or variable cost. Thus, diagramatically

$$\begin{tikzcd}
 & X_1 \arrow[dr, "f"] & \\
b \arrow[ur, "g"] \arrow[dr, "g"'] & & TR(or\ TVC) \\
 & X_2 \arrow[ur, "f"'] &
\end{tikzcd}$$

Increasing b relaxes the constraint, g, causing X_1 and X_2, through f, to increase TR (or TVC). In the maximizing example, we would have

$$\lambda = \frac{dTR}{dTVC}$$

the addition to total revenue resulting from an addition to total variable cost while in the minimizing example, we would have

$$\lambda = \frac{dTVC}{dY}$$

the addition to total variable cost resulting from an addition to output. Therefore, on the expansion path, for minimization example

$$\lambda = \frac{dTVC}{dY} = \text{Marginal cost}$$

Finally, the fact that λ is the total derivative of f with respect to b can be derived. Consider the objective and constraint functions simultaneously

$$Y = f(X_1, X_2)$$
$$b = g(X_1, X_2)$$

where X_1 and X_2 are regarded as functions of b. Then by differentiating implicitly

$$\frac{dY}{db} = \frac{\partial f}{\partial X_1}\frac{dX_1}{db} + \frac{\partial f}{\partial X_2}\cdot\frac{dX_2}{db}$$

$$1 = \frac{\partial g}{\partial X_1}\frac{dX_1}{db} + \frac{\partial g}{\partial X_2}\cdot\frac{dX_2}{db}$$

Multiplying the second equation by λ and subtracting it from the first equation will yield an equation in which most terms will vanish at the point of the constrainted optimum, yielding the result.

The dual of the Lagrangean. We discussed the dual of the linear programming problem in detail. The dual solution exists for all

problems that can be solved using the Lagrangean technique. We will state the concept of the Lagrangean dual without proof; the proof is complex and beyond the scope of this discussion.

Assume a particular point (X_1°, X_2°) yields the (constrained) maximum of the objective function $f(X_1, X_2)$ subject to the constraint $g(X_1, X_2) = b$. This is the primal. Then the dual problem can be stated as: The point $(X_1^\circ, X_2^\circ, \lambda^\circ)$ gives the solution that minimizes $F(X_1, X_2, \lambda)$ subject to

$$\frac{\partial f}{\partial X_1} - \lambda^\circ \frac{\partial g}{\partial X_1} = 0$$

$$\frac{\partial f}{\partial X_2} - \lambda^\circ \frac{\partial g}{\partial X_2} = 0$$

where all derivatives are evaluated at (X_1°, X_2°) and

$$\lambda^\circ = \frac{\dfrac{\partial f(X_1^\circ, X_2^\circ)}{\partial X_1}}{\dfrac{\partial g(X_1^\circ, X_2^\circ)}{\partial X_2}}$$

Many problems in economics have duals. For example, variable cost can be minimized subject to a particular output (total revenue) level. Or, output (total revenue) can be maximized subject to a specific level of variable cost. The answer obtained will be identical if the constraint levels are consistent. In the examples above, it was shown that the answer would be the same if, when maximizing, variable cost was limited to \$20.80 or, when minimizing, output was limited to 105. This example of the dual was worked above and will not be repeated here.

It is useful to illustrate the dual by using a linear programming problem from Chapter 9. One problem was to maximize $3Y_1 + 2Y_2$ subject to

$$Y_1 + 0.5\ Y_2 \leqq 4$$

$$Y_1 + Y_2 \leqq 5$$

When set up as a Lagrangean function, we will get:

$$F(Y_1, Y_2, \lambda_1, \lambda_2) = 3Y_1 + 2Y_2 - \lambda_1(Y_1 + 0.5Y_2 - 4) - \lambda_2(Y_1 + Y_2 - 5)$$

which has the first order partial derivatives

$$\frac{\partial F}{\partial Y_1} = 3 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial F}{\partial Y} = 2 - 0.5\,\lambda_1 - \lambda_2 = 0$$

$$\frac{\partial F}{\partial \lambda_1} = Y_1 + 0.5Y_2 - 4 = 0$$

$$\frac{\partial F}{\partial \lambda_2} = Y_1 + Y_2 - 5 = 0$$

Solving the third and fourth equations simultaneously gives $Y_1 = 3$ and $Y_2 = 2$ which gives a total profit of \$13 (See Figure 9-5A.) Solution of the first two equations gives the shadow prices of $\lambda_1 = \$2$ and $\lambda_2 = \$1$.

The dual of this problem would be to minimize $F(Y_1, Y_2, \lambda_1, \lambda_2)$ subject to $3 - \lambda_1 - \lambda_2 = 0$ and $2 - 0.5\lambda_1 - \lambda_2 = 0$. However, the function $F(Y_1, Y_2, \lambda_1, \lambda_2)$ will simplify to $4\lambda_1 + 5\lambda_2$ so the problem can be restated

$$\begin{aligned} \text{minimize} \quad & 4\lambda_1 + 5\lambda_2 \\ \text{subject to} \quad & \lambda_1 + \lambda_2 = 3 \\ & 0.5\lambda_1 + \lambda_2 = 2 \end{aligned}$$

but this is the dual problem stated in Chapter 9.

This example is very special. In this application, we knew that the solution would be positive and the equalities would hold at the point of solution. In general, linear programming problems cannot be cast in the Lagrangean mold because of the possible existence of corner solutions and even negative solutions. That is, the Lagrangean technique does not in general permit corner or boundary solutions and does permit negative solutions. Throughout this section the examples have been carefully tailored to avoid violating these assumptions. But, many examples could be chosen that do in fact violate these conditions—conditions that are necessary in economics. In the next section we present a glimpse of the more general techniques.

Inequality Constraints

We will now incorporate the nonnegativity requirements and inequality constraints into our analysis. First, we will consider maximizing a function of one variable, $Y = f(X)$ subject to $X \geqq 0$. If the function f takes a maximum at a point, say, X_0, where $X_0 \geqq 0$, then it will be true (necessary) that

$$\frac{df(X_0)}{dX} \leqq 0 \quad \text{and} \quad \frac{df(X_0)}{dX} \cdot X_0 = 0$$

That is, either the first derivative is zero at X_0, or X_0 is zero or both happen. This can be demonstrated using geometry. Refer to Figure II-1; in each case, we seek the maximum of the function for the set of points, $X \geqq 0$. In Figure II-1A, the function is linear and decreases throughout the feasible set—therefore, $df(X_1)/dX < 0$ and $X_0 = 0$. In B, the function does take a maximum but outside the

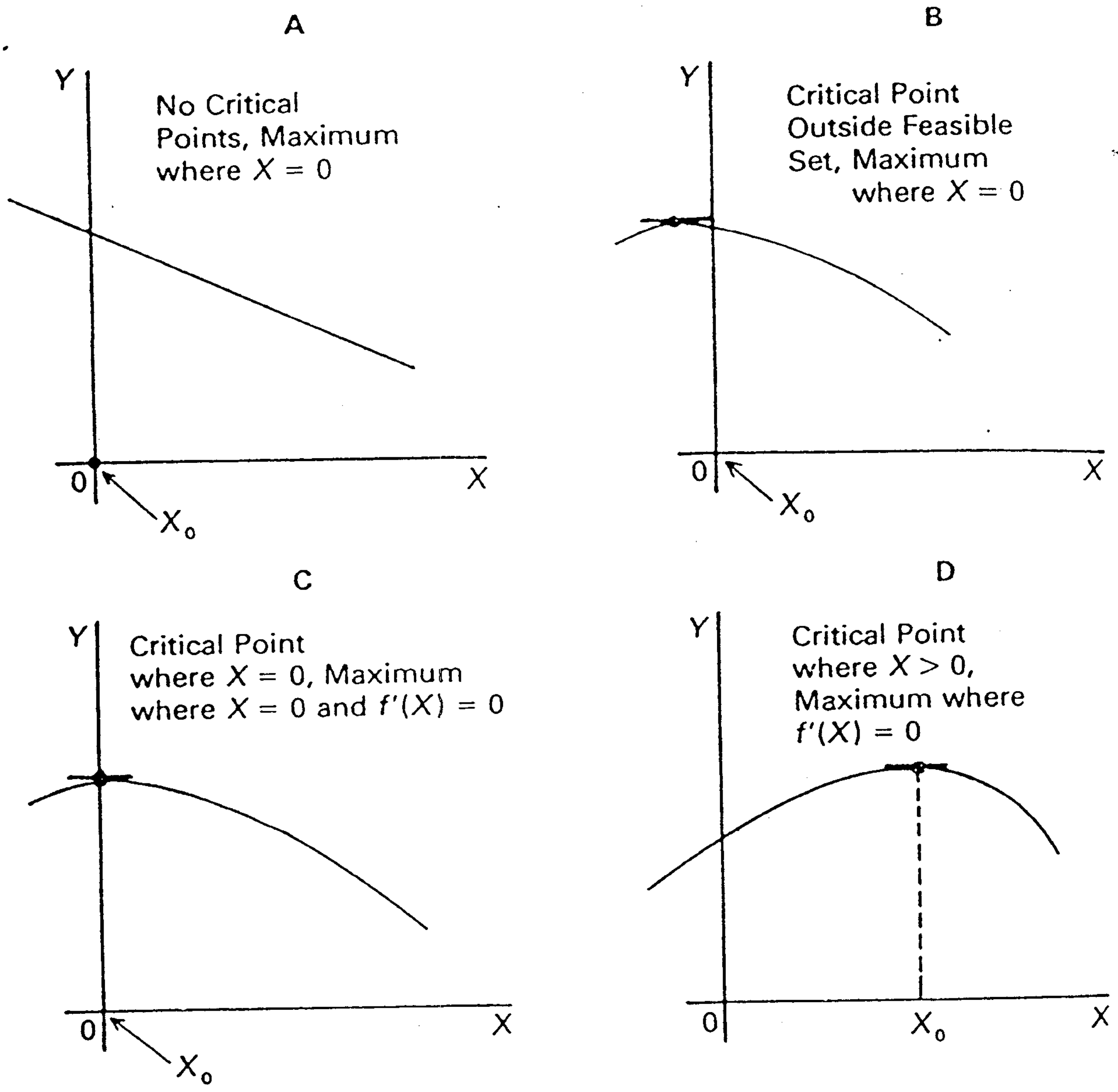

FIGURE II-1

FINDING THE MAXIMUM OF A FUNCTION, $Y = f(X)$, WHEN $X \geqq 0$

feasible set, so again at X_o, $df(X_o)/dX < 0$ and $X_o = 0$. In Figure II-1C, the function's critical point is at zero, so it is true that $df(X_o)/dX = 0$ and $X_o = 0$. In case D, the critical point is within the feasible set so that $df(X_o)/dX = 0$ even though $X_o > 0$. In each of the four cases, the necessary conditions stated above will hold.

The one variable case can be immediately generalized to two or more variables. Assume now that the function $Y = f(X_1, X_2)$ has a maximum subject to $X_1 \geqq 0$ and $X_2 \geqq 0$. Then again it will be true that

$$\frac{\partial f(X_1^\circ, X_2^\circ)}{\partial X_1} \leqq 0$$

$$\frac{\partial f(X_1^\circ, X_2^\circ)}{\partial X_2} \leqq 0$$

and

$$\frac{\partial f(X_1^\circ, X_2^\circ)}{\partial X_1} \cdot X_1^\circ = 0$$

$$\frac{\partial f(X_1^\circ, X_2^\circ)}{\partial X_2} \cdot X_2^\circ = 0$$

Again, these conditions are based upon the same reasoning presented for a function of one variable. If the function has a critical point (assumed here to be a maximum) where X_1 and X_2 are positive, the first derivatives will vanish. If the maximum is on a boundary (either axis) or in the second or fourth quadrants, the first derivative in question will be zero or negative but in either case the value of the variable, X_1° or X_2°, will be zero. If the maximum occurs at the origin (or in the third quadrant which is outside the feasible set) then $X_1^\circ = X_2^\circ = 0$ and both first partial derivatives will be zero (or negative).

We now complete this example by adding the inequality constraint. The problem now is to maximize an objective function $Y = f(X_1, X_2)$ subject to $g(X_1, X_2) \leqq b$ and $X_1 \geqq 0$, $X_2 \geqq 0$. The functions f and g can be linear or nonlinear; there are no constraints on their form. They must have derivatives, however. This problem satisfies the needs of the linear programming formulation, but it is much more general. When the functions f and g are not linear, these problems are called, not surprisingly, nonlinear programming problems.

The theorem starts with the Lagrangean function

$$F(X_1, X_2, \lambda) = f(X_1, X_2) + \lambda(b - g(X_1, X_2))$$

and the function f is assumed to take a maximum subject to the constraint g at the point $X^\circ = (X_1^\circ, X_2^\circ)$. The Lagrangean extremal would be at $P^\circ = (X_1^\circ, X_2^\circ, \lambda^\circ)$. Then there will exist a λ° such that the following conditions will always hold

Condition 1 $$\frac{\partial F(P^\circ)}{\partial X_1} \leqq 0 \quad \text{or} \quad \lambda_{x_1} = \frac{\dfrac{\partial f(X^\circ)}{\partial X_1}}{\dfrac{\partial g(X^\circ)}{\partial X_1}} \leqq \lambda^\circ$$

$$\frac{\partial F(P^\circ)}{\partial X_2} \leqq 0 \quad \text{or} \quad \lambda_{x_2} = \frac{\dfrac{\partial f(X^\circ)}{\partial X_2}}{\dfrac{\partial g(X^\circ)}{\partial X_2}} \leqq \lambda^\circ$$

Condition 2 $$\frac{\partial F(P^\circ)}{\partial X_1} \cdot X_1^\circ + \frac{\partial F(P^\circ)}{\partial X_2} \cdot X_2^\circ = 0$$

$$\text{Condition 3} \quad \frac{\partial F(P^\circ)}{\partial \lambda} = b - g(X^\circ) \geqq 0$$

$$\text{Condition 4} \quad \frac{\partial F(P^\circ)}{\partial \lambda} \cdot \lambda^\circ = [b - g(X^\circ)] \cdot \lambda^\circ = 0$$

Conditions 1 and 2 are based upon the nonnegativity restrictions already developed. Only the addition of λ°, the value of the shadow price at the optimum, is new. It is instructive to work out all possible cases to insure condition 2 is always zero. Condition 3 follows because the extreme value of f will either fall on the constraint ($g(X^\circ) = b$) or within it ($b > g(X^\circ)$). Condition 4 says that either the constraint is fulfilled ($g(X^\circ) = b$) or that λ° is zero. If the extreme occurs where the constraint is not fulfilled—at a point interior to the feasible set—the shadow price is zero. The objective function cannot be increased by increasing b because f is already a maximum. There is a surplus of whatever is represented by the constraint.

Finally, the optimal value of λ, called λ°, must be chosen to fulfill conditions 3 and 4. λ° will always be chosen to be the maximum of three values.

$$\lambda^\circ = \text{maximum} \left[\frac{\dfrac{\partial f(X^\circ)}{\partial X_1}}{\dfrac{\partial g(X^\circ)}{\partial X_1}}, \frac{\dfrac{\partial f(X^\circ)}{\partial X_2}}{\dfrac{\partial g(X^\circ)}{\partial X_2}}, 0 \right]$$

That is, when the solution is found, X_1 will have the largest shadow price, or X_2 will have the largest shadow price (not ruling out, of course, the case where they are equal), or the shadow price will be zero.

The best way to illustrate all this is to work an elementary example. Consider an objective function of the form

$$Y = 8X_1 - X_1^2 + 12X_2 - X_2^2 = f(X_1, X_2)$$

with the constraint

$$5X_1 + X_2 = 5 = g(X_1, X_2)$$

and X_1 and X_2 are nonnegative. We can regard Y as total revenue and X_1 and X_2 as inputs. The objective function is quadratic and the constraint is linear; we have what is often called a quadratic programming problem. Nonlinear programming problems do not have simple but general algorithms for solution such as the Simplex solution for linear programming. The solution is found by analysis of the functions involved, and each situation may be different. In this case, it makes sense to begin by analyzing the properties of the objective function, f.

The first partial derivatives of f are

$$\frac{\partial f}{\partial X_1} = 8 - 2X_1$$

$$\frac{\partial f}{\partial X_2} = 12 - 2X_2$$

Setting these equal to zero and solving gives $X_1 = 4$ and $X_2 = 6$. The function takes an unconstrained maximum of $Y = 52$ at this point. The derivative for X_1 is positive when $X_1 = 0$ so the function is increasing in the X_1 direction (that is, when the X_2 axis is crossed on any line parallel to the X_1 axis). The same is true for X_2. Ridge lines for this function occur when $X_1 = 4$ and $X_2 = 6$. Thus, the constrained maximum we seek will fall in the set of points enclosed by

$$0 \leqq X_1 \leqq 4$$

$$0 \leqq X_2 \leqq 6$$

This set of points is depicted in Figure II-2.

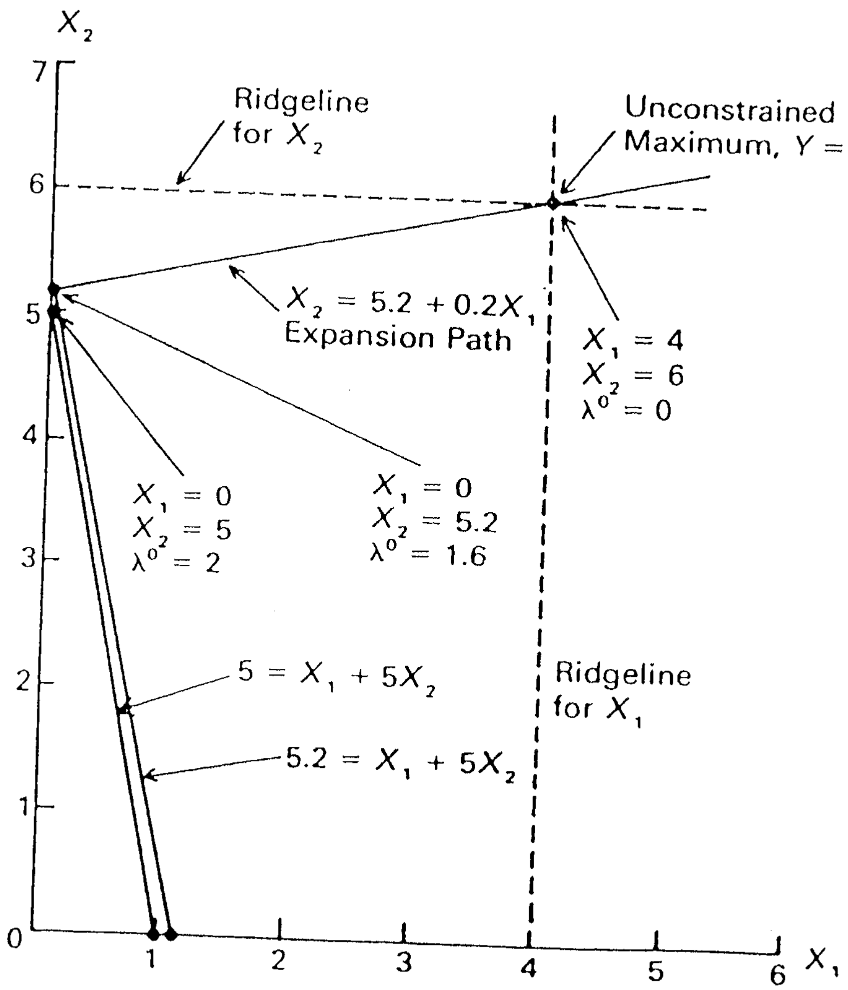

FIGURE II–2

A CONSTRAINED MAXIMIZATION PROBLEM

The constraint can be expressed as

$$X_2 = 5 - 5X_1$$

and when graphed appears as a straight line between the points (X_1, X_2) = (0,5) and (1,0). The unconstrained maximum of f falls outside the constraint. Therefore, the constraint will be satisfied at the constrained maximum and could be on the constraint line within the first quadrant or on the endpoints. To solve, we can substitute the constraint, g, directly into the function, f, as follows

$$Y_c = 8X_1 - X_1^2 + 12(5 - 5X_1) - (5 - 5X_1)^2$$
$$Y_c = 35 - 2X_1 - 26\,X_1^2$$

for $0 \leqq X_1 \leqq 1$. That is, this is the value of function above the constraint only. Visual examination of this function shows that it is maximized within the set when $X_1 = 0$. X_2 will be 5 and output will be 35. This is a corner solution. Examining the conditions which should hold when the isoquant for $Y = 35$ is tangent to the isocost line we find

$$\frac{MPP_{x1}}{MPP_{x2}} = \frac{P_{x1}}{P_{x2}}$$
$$\frac{8 - 2X_1}{12 - 2X_2} = \frac{8}{2} \neq \frac{5}{1}$$

Thus the tangency condition is not fulfilled. By examining the necessary conditions for a constrained maximum we can determine λ°.

$$F(X_1, X_2, \lambda) = (8X_1 - X_1^2 + 12X_2 - X_2^2) + \lambda(5 - 5X_1 - X_2)$$

so that

$$\frac{\partial F}{\partial X_1} = 8 - 2X_1 - 5\lambda \leqq 0$$
$$\frac{\partial F}{\partial X_2} = 12 - 2X_2 - \lambda \leqq 0$$
$$\frac{\partial F}{\partial \lambda} = 5 - 5X_1 - X_2 \geqq 0$$

Substituting the optimal values of X_1 and X_2 in the first two equations yields (ignoring the inequalities)

$$\lambda_{x1} = \frac{\dfrac{\partial f}{\partial X_1}}{\dfrac{\partial g}{\partial X_1}} = \frac{8 - 2X_1}{5} = \frac{8}{5}$$

or

$$\lambda_{x_2} = \frac{\dfrac{\partial f}{\partial X_2}}{\dfrac{\partial g}{\partial X_2}} = \frac{12 - 2X_2}{1} = \frac{2}{1} = 2$$

We chose $\lambda^\circ = \max \{1.6, 2, 0\}$. But since the constraint is satisfied, λ° cannot be zero. If we chose λ° to be 1.6, the second partial derivative is not satisfied. Thus, $\lambda_o = 2$. The largest shadow price is associated with X_2. Increasing X_2 will increase the objective function more than increasing X_1. We will not review them, but the student should check that $(X_1^\circ, X_2^\circ, \lambda^\circ) = (0, 5, 2)$ do satisfy all the necessary conditions for the constrained maximum.

From the standpoint of economic theory, the expansion path for this example starts at the origin and extends out the X_2 axis. After $X_2 = 5.2$, the expansion path becomes $X_2 = 5.2 + 0.2X_1$. Thus, 5.2 units of X_2 will be used before X_1 will be used.

Suppose the constraint is increased to 5.2 (b is increased to 5.2). The constrainted maximum will be at the point $(X_1, X_2) = (0, 5.2)$. What will λ° be? Again, using the equations above

$$\lambda_{x_1} = \frac{8 - 2X_1}{5} = \frac{8}{5} = 1.6$$

$$\lambda_{x_2} = \frac{12 - 2X_2}{1} = \frac{1.6}{1} = 1.6$$

and the constraint is again satisfied. λ° is chosen to be 1.6. This is again a corner solution, but we have moved out the expansion path to the point where X_1 now enters into the least cost combination. Another way to explain it is that as the use of X_2 increased, its marginal productivity dropped until it becomes feasible to use the high priced input, X_1. Substituting into the tangency conditions for a least cost combination we find

$$\frac{MPP_{x_1}}{MPP_{x_2}} = \frac{8 - 2X_1}{12 - 2X_2} = \frac{8}{1.6} = \frac{5}{1} = \frac{P_{x_1}}{P_{x_2}}$$

They are satisfied.

As the constraint is increased further, the constrained maximum will move along the expansion path, and it will always be true that $\lambda_{x_1} = \lambda_{x_2} = \lambda^\circ > 0$. Suppose that b were increased to 26, then the constrained optimum would be $(X_1, X_2) = (4, 6)$ and would coincide with the unconstrained optimum. At that point

$$\lambda_{x_1} = \frac{8 - 2X_1}{5} = \frac{8 - 8}{5} = 0$$

$$\lambda_{x_2} = \frac{12 - 2X_2}{1} = \frac{0}{1} = 0$$

and the constraint is satisfied. λ° would be chosen to be zero.

Finally, if the constraint value is increased past 26, say 30, the constrained optimum remains at $(X_1, X_2) = (4,6)$. The constraint is never satisfied, $b > g(X_1, X_2)$ so that $\lambda^\circ = 0$. The shadow prices of the inputs are zero. However, all the necessary conditions for the constrained optimum are satisfied. For example

Condition 1 $$\frac{\partial F}{\partial X_1} = 8 - 2X_1 - 5\lambda = 8 - 2 \cdot 4 - 5 \cdot 0 \leqq 0$$

$$\frac{\partial F}{\partial X_2} = 12 - 2X_2 - \lambda = 12 - 2 \cdot 6 - 0 \leqq 0$$

Condition 2 $$\frac{\partial F}{\partial X_1} \cdot X_1 + \cdot \frac{\partial F}{\partial X_2} \cdot X_2 = 0 \cdot 4 + 0 \cdot 6 = 0$$

Condition 3 $$\frac{\partial F}{\partial \lambda} = 30 - 5X_1 - X_2 = 30 - 5 \cdot 4 - 6 = 4 \geq 0$$

Condition 4 $$\frac{\partial F}{\partial \lambda} \cdot \lambda = [30 - 26] \cdot 0 = 0$$

We have just worked an example of a nonlinear programming problem in detail. Although it is a special case, it does serve to illustrate many of the general principles. In general, neither f or g have to be linear but can assume any forms. The theory remains the same, but the search for the solution becomes much more difficult. Fortunately, computer software abounds with algorithms designed to solve special problems.

ENDNOTES

1. We will not consider the sufficient conditions for a maximum or a minimum in this section. The sufficient conditions require more mathematical background then we will utilize here. Examples chosen will always meet the sufficient conditions.
2. Although total revenue is a function of both inputs, this "reduced" function can be referred to as a parametric representation where X_2 is the parameter. The equation could also be written as a function of the parameter t where $X_1 = 10.4 - 1.5t$ and $X_2 = t$.

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